Spectral filtering for hierarchical search of periodic sources

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“Old” hierarchical method

• Divide the data in (interlaced) chunks; the length is such that the signal remains inside one frequency bin

• Do the FFT of the chunks; the archived collection of these FFT is the SFDB

• Do the first “incoherent step” (Hough or Radon transform) and take candidates to “follow”

• Do the first “coherent step”, following up candidates with longer (about 16 times) “corrected” FFTs, obtaining a refined SFDB (on the fly)

• Repeat the preceding two step, until we arrive at the full resolution
# The 4 SFDB bands

<table>
<thead>
<tr>
<th></th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Max frequency of the band</strong></td>
<td>2000</td>
<td>500</td>
<td>125</td>
<td>31.25</td>
</tr>
<tr>
<td><em>(Nyquist frequency)</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Observed frequency bands</strong></td>
<td>1500</td>
<td>375</td>
<td>93.75</td>
<td>23.438</td>
</tr>
<tr>
<td><strong>Max duration for an FFT (s)</strong></td>
<td>2445</td>
<td>4891</td>
<td>9782</td>
<td>19565</td>
</tr>
<tr>
<td><strong>Length of the FFTs</strong></td>
<td>4194304</td>
<td>4194304</td>
<td>2097152</td>
<td>1048576</td>
</tr>
<tr>
<td><strong>FFT duration (s)</strong></td>
<td>1048</td>
<td>4194</td>
<td>8388</td>
<td>16777</td>
</tr>
<tr>
<td><strong>Number of FFTs</strong></td>
<td>20063</td>
<td>5015</td>
<td>2508</td>
<td>1254</td>
</tr>
<tr>
<td><strong>SFDB storage (GB; one year)</strong></td>
<td>510</td>
<td>130</td>
<td>33</td>
<td>9</td>
</tr>
</tbody>
</table>
## Scheme of the detection

\[ T_{\text{OBS}} = 4 \text{ months} \quad T_{\text{FFT}} = 3355 \text{ s} \]

<table>
<thead>
<tr>
<th>step</th>
<th>( T_{\text{FFT}} )</th>
<th>N points</th>
<th>SNR (linear)</th>
<th>CR</th>
<th>Normal probability</th>
<th>Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(~1 \text{ h})</td>
<td>1.5 e15</td>
<td>2</td>
<td>4</td>
<td>3.1 e-5</td>
<td>5 e10</td>
</tr>
<tr>
<td>2</td>
<td>15 h</td>
<td>9.8 e19</td>
<td>4</td>
<td>16</td>
<td>~1 e-55</td>
<td>1 e-35</td>
</tr>
<tr>
<td>3</td>
<td>10 d</td>
<td>6.4 e24</td>
<td>8</td>
<td>64</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>(~4 \text{ m})</td>
<td>4.2 e29</td>
<td>~16</td>
<td>~256</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Candidate sources

The result of the first incoherent step is a list of candidates (for example, $10^9$ candidates).

Each candidate has a set of parameters:

- the frequency at a certain epoch
- the position in the sky
- 1~2 spin-down parameters
Coherent steps

With the coherent step we partially correct the frequency shift due to the Doppler effect and to the spin-down. Then we can do longer FFTs, and so we can have a more refined time-frequency map.

This step is done only on “candidate sources”, survived to the preceding incoherent step.
Coherent follow-up

- Extract the band containing the candidate frequency (with a width of the maximum Doppler effect plus the possible intrinsic frequency shift)
- Obtain the time-domain analytic signal for this band (it is a complex time series with low sampling time (lower than 1 Hz))
- Multiply the analytic signal samples for \( e^{-j\Delta\omega_{D}t_i} \), where \( t_i \) is the time of the sample, and \( \Delta\omega_{D} \) is the correction of the Doppler shift and of the spin-down.
- Create a new (partial) FFT data base now with higher length (dependent on the precision of the correction) and the relative time-frequency spectrum and peak map. Note that we are now interested to a very narrow band, much lower than the Doppler band.
- Do the Hough transform on this (new incoherent step).
Problems in the second step

In the first step the sampling time (always < 6 hours) is such that features with frequency of the order of the sidereal frequency (~1/86000 Hz) are not resolved.

With the second step, two effects:

- the amplitude modulation of the signal, due to the radiation pattern of the antenna
- the change of the polarization rotation due to the Earth rotation

spreads the signal in 5 bands.

So in the second step the signal (and the SNR) can be lower than in the first.
Simplified case:

Virgo is displaced to the terrestrial North Pole and the pulsar is at the celestial North Pole.

The inclination of the pulsar can be any.
Simplified case
in red the original frequency

Circular polarization

Linear polarization

Mixed polarization

Circular polarization (reverse)
The general case

The response of a GW detector can be computed as the product of the two tensors $M$ describing the wave and $D$ describing the detector

$$M_j D_j$$

where $D$ is, for example,

for a bar

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

for an interferometer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$M$ is highly more complex.

4 The $M$ matrix

The response of a generic gravitational wave detector can be expressed through symmetric and traceless matrix, which for a $+1$ wave has the following components (in the detector reference frame):

$$M_{11} = \cos^2 \Psi \cos^2 \phi - 1 - \cos \Psi \cos \phi + \sin \Psi \sin \phi - \cos^2 \Psi \cos^2 \phi$$

$$M_{12} = \cos^2 \Psi \sin^2 \phi$$

$$M_{13} = -\cos^2 \Psi \sin \phi \cos \phi$$

$$M_{22} = -\cos^2 \Psi \sin \phi \sin \phi$$

$$M_{23} = \cos^2 \Psi \sin \phi \sin \phi$$

$$M_{33} = \cos^2 \Psi \sin^2 \phi$$

These quantities can be expressed in terms of combinations of elements of the matrix $R_{det}$:

$$M_{11} = (R_{det}^{ij})^2 - (R_{det}^{ij})^2$$

$$M_{12} = R_{det}^{ij} R_{det}^{ij} - R_{det}^{ij} R_{det}^{ij}$$

$$M_{13} = R_{det}^{ij} R_{det}^{ij} - R_{det}^{ij} R_{det}^{ij}$$

$$M_{22} = -(R_{det}^{ij})^2 + (R_{det}^{ij})^2$$

$$M_{23} = R_{det}^{ij} R_{det}^{ij} - R_{det}^{ij} R_{det}^{ij}$$

$$M_{33} = (R_{det}^{ij})^2 - (R_{det}^{ij})^2$$

Then, using Eq. (8) we can calculate the matrix $M$ in the celestial frame and then obtain the dependence on the (local) sidereal time. Let us introduce the detector azimuth, $\alpha$, defined as the angle between the south direction and the detector x-axis. The relation with the angle $\beta$ used before is $\alpha + \beta = \pi$. After heavy algebraic manipulations we arrive at the following expressions:

$$M_{ij} = a_{ij} \cdot \sin(2\theta) + b_{ij} \cdot \cos(2\theta)$$

where

$$a_{11} = \frac{1}{2} (-\cos(2\alpha) + 1) \cos(\beta) \sin(2\beta) \sin(\alpha + \beta) + \frac{1}{4} (1 - \cos(2\alpha)) \cos(2\beta) = \frac{3}{16} \cos(2\beta)$$

$$b_{11} = \frac{1}{16} \cos(2\beta)$$

$$a_{12} = \frac{1}{2} (-\cos(2\beta) + 1) \sin(2\beta) \sin(2\beta) \sin(\alpha + \beta) + \frac{1}{16} (1 - \cos(2\beta)) \cos(2\beta)$$

$$b_{12} = \frac{1}{8} \cos(2\beta)$$
\[
\begin{align*}
\alpha_{22} &= \frac{1}{16} (3 - \cos(2\delta)) \cos(\alpha - \Omega) \sin(\delta) \sin(2\alpha - 2\Omega) + \\
&\quad \frac{1}{4} \cos(\alpha - \Omega) \cos(\delta) \cos(\lambda) \sin(2\alpha) + \\
&\quad \sin(\alpha - \Omega) \cos(\alpha - \Omega) \cos(\delta) \sin(2\lambda) \cos^2(\alpha) + \\
&\quad \cos(2\alpha - 2\Omega) \sin(\delta) \sin(\lambda) \sin(2\alpha) \\
\beta_{22} &= \frac{1}{4} (1 - 3 + \cos(2\delta)) \sin(\lambda) \sin(2\alpha - 2\Omega) + \\
&\quad \frac{1}{8} (3 - \cos(2\delta)) \frac{1}{4} \left[ 1 - \cos(2\delta) \right] \left( 1 + \cos(2\alpha) \right) \cos(2\alpha - 2\Omega) + \\
&\quad \frac{1}{4} \cos(\alpha - \Omega) \sin(\delta) \sin(2\alpha) \cos^2(\delta) - \\
&\quad \frac{1}{2} \sin(\alpha - \Omega) \sin(2\delta) \cos(\lambda) \sin(2\alpha) \\
\alpha_{13} &= -\frac{1}{3} \sin(2\alpha - 2\Omega) \sin(\delta) \sin(\alpha - \Omega) \sin(2\lambda) + \\
&\quad \cos(2\alpha - 2\Omega) \sin(\delta) \sin(\alpha) \cos(\lambda) - \\
&\quad \cos(\alpha - \Omega) \cos(\delta) \sin(\alpha) \cos(\lambda) - \\
&\quad \sin(\alpha - \Omega) \cos(\delta) \sin(\lambda) \cos(2\alpha) \\
\beta_{13} &= \frac{1}{4} (1 - 3 + \cos(2\delta)) \cos(\lambda) \cos(\alpha - \Omega) \sin(2\alpha - 2\Omega) + \\
&\quad \frac{1}{2} \sin(\alpha - \Omega) \sin(2\delta) \sin(\lambda) \sin(2\alpha) + \\
&\quad \frac{1}{2} \cos(\alpha - \Omega) \sin(\lambda) \sin(2\alpha) \cos(\lambda) - \\
&\quad \frac{3}{4} \left( 1 + \cos(2\delta) \right) \sin(2\lambda) \cos(\alpha) \\
\alpha_{23} &= -\cos(\alpha - \Omega) \sin(\alpha) \cos(\delta) \sin(\lambda) + \\
&\quad \cos(2\alpha - 2\Omega) \sin(\delta) \sin(\alpha) \cos(\lambda) + \\
&\quad \frac{1}{2} \sin(2\alpha - 2\Omega) \cos(2\alpha - 2\Omega) \sin(\delta) \sin(\alpha) + \\
&\quad \sin(\alpha - \Omega) \cos(\alpha - \Omega) \cos(\delta) \cos(\lambda) \\
\beta_{23} &= \frac{1}{4} (1 - 3 + \cos(2\delta)) \cos(\lambda) \sin(\alpha) \sin(2\alpha - 2\Omega) + \\
&\quad \frac{1}{4} (3 - \cos(2\delta)) \sin(\lambda) \cos(2\alpha - 2\Omega) + \\
&\quad \frac{1}{2} \sin(\alpha - \Omega) \sin(2\delta) \sin(\alpha) \sin(\lambda) + \\
&\quad \frac{1}{2} \cos(\alpha - \Omega) \sin(2\delta) \cos(\alpha) \sin(\lambda) - \\
&\quad \frac{3}{8} \left( 1 + \cos(2\delta) \right) \sin(2\lambda) \cos(\alpha) \\
\alpha_{32} &= \frac{1}{4} (1 + \cos(2\delta)) \sin(\lambda) \sin(2\alpha - 2\Omega) + \\
&\quad \cos(\alpha - \Omega) \cos(\delta) \sin(\lambda) \cos(2\alpha) + \\
&\quad \cos(2\alpha - 2\Omega) \sin(\delta) \sin(\lambda) \cos(2\alpha) - \\
&\quad \frac{1}{2} \sin(\alpha - \Omega) \cos(\lambda) \sin(2\delta) \sin(2\alpha) \\
\beta_{32} &= \frac{1}{4} (1 - 3 + \cos(2\delta)) \sin(\lambda) \cos(2\alpha - 2\Omega) + \\
&\quad \frac{1}{4} (3 - \cos(2\delta)) \sin(\lambda) \cos(2\alpha - 2\Omega) - \\
&\quad \frac{1}{2} \sin(2\alpha - 2\Omega) \sin(\delta) \sin(\lambda) + \\
&\quad \frac{1}{2} \cos(\alpha - \Omega) \sin(2\delta) \cos(\lambda) - \\
&\quad \frac{3}{8} \left( \frac{3}{2} \sin^2(\lambda) - \cos(2\lambda) \cos^2(\delta) \right) \\
\end{align*}
\]

The corresponding quantities for a 'z'-polarization wave can be easily obtained from the above formulae with the replacement \(\varphi' = \varphi' + \frac{\pi}{2} \).

The beam-pattern functions of a specific kind of detector can be expressed as a suitable combination of elements of the matrix \(M\). For instance, for an interferometric detector:

\[
F_+ = \frac{M_{11} - M_{22}}{2}
\]

and the same for \(F_\times\), using the rotated \(M\) matrix.
### Some cases

If the original angular frequency is at $\omega_0$, the power goes in five bands at

$$\omega_0-2\Omega \quad \omega_0-\Omega \quad \omega_0 \quad \omega_0+\Omega \quad \omega_0+2\Omega$$

depending on the **fixed** parameters of the position of the antenna and the position of the source and on the **variable** parameters describing the polarization:

- $\varepsilon$, the percentage of linear polarization
- $\psi$, the polarization angle (for lin. pol.)
- the rotation direction (for circ. pol.)

| $\Psi = 0$, $\varepsilon = 0\, 0.2\, 0.4\, 0.6\, 0.8\, 1$, lines $-2\,-1\,\,0\,\,1\,\,2$ |
|---------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| **Delta = 90**            | 0.0000              | 0.0001              | 0.0002              | 0.0008              | 0.9990              |
|                           | 0.0554              | 0.0001              | 0.0002              | 0.0008              | 0.9436              |
|                           | 0.1249              | 0.0001              | 0.0002              | 0.0007              | 0.8742              |
|                           | 0.2142              | 0.0001              | 0.0001              | 0.0006              | 0.7849              |
|                           | 0.3333              | 0.0001              | 0.0001              | 0.0006              | 0.6659              |
|                           | 0.5002              | 0.0001              | 0.0001              | 0.0004              | 0.4992              |
| **Delta = 70**            | 0.0000              | 0.0001              | 0.0042              | 0.0543              | 0.9414              |
|                           | 0.0554              | 0.0033              | 0.0037              | 0.0515              | 0.8893              |
|                           | 0.1175              | 0.0073              | 0.0032              | 0.0479              | 0.8241              |
|                           | 0.2016              | 0.0124              | 0.0024              | 0.0434              | 0.7401              |
|                           | 0.3139              | 0.0192              | 0.0015              | 0.0374              | 0.6281              |
|                           | 0.4715              | 0.0286              | 0.0001              | 0.0289              | 0.4709              |
| **Delta = 50**            | 0.0003              | 0.0032              | 0.0596              | 0.1833              | 0.7536              |
|                           | 0.0408              | 0.0160              | 0.0532              | 0.1768              | 0.7132              |
|                           | 0.0920              | 0.0321              | 0.0452              | 0.1685              | 0.6623              |
|                           | 0.1586              | 0.0531              | 0.0347              | 0.1577              | 0.5960              |
|                           | 0.2488              | 0.0814              | 0.0205              | 0.1430              | 0.5063              |
|                           | 0.3780              | 0.1220              | 0.0001              | 0.1219              | 0.3779              |
Two cases
(Actually Virgo in Cascina and pulsar in GC)

Linear polarization, $\psi=0$  
Circular polarization
Matched filter on the spectrum

- In the first step the peaks of the time-frequency spectrum are directly spotted (after a particular equalization).

- In the second step the spectra should be filtered with a battery of matched filters depending on the linear-circular polarization mixture and the declination of the source. Moreover the second step should have an higher frequency enhancement (for example 64 instead of 16).
The spectral filter

Let

\[ w_{-2} \quad w_{-1} \quad w_0 \quad w_1 \quad w_2 \]

be five numbers proportional to the power content of the five bands, with

\[ \sum_{k=-2}^{2} w_k^2 = 1 \]

we build the matched filter on the spectrum \( S(\omega) \)

\[ y(\omega) = \sum_{k=-2}^{2} w_k \cdot S(\omega + k \cdot \Omega) \]

where \( \Omega \) is the sidereal angular frequency. Because \( \Omega \) may correspond to a non-integer number of frequency bins, we implement the filter in the delay domain (and the spectra are windowed and with over-resolution). Let \( R(\tau) \) be the Fourier transform of \( S(\omega) \), we compute

\[ Y(\tau) = R(\tau) \cdot \sum_{k=-2}^{2} w_k e^{-j\tau k\Omega} \]

and then obtain the filtered spectrum as the inverse transform of \( Y(\tau) \).
The filter performance

Because of the chosen normalization, in absence of noise, the gain of the filter to the
proper signal is unitary. The noise distribution is about a $\chi^2$-oidal distribution (with a
linear transformation of the abscissa and possibly a not integer d.o.f.). If the input
spectrum is distributed exponentially with $\mu=\sigma=1$, we have

$$\sigma_y^2 = \sum_{k=-2}^{2} w_k^2 = 1 \quad \text{and} \quad \mu_y = \sum_{k=-2}^{2} w_k$$

and an equivalent number of degrees of freedom given by

$$\tilde{N} = \frac{2\mu_y^2}{\sigma_y^2} = 2 \cdot \left( \sum_{k=-2}^{2} w_k \right)^2$$

$\tilde{N}$ is between 2 and 10 and we can put

$$\mu_y = \sqrt{\frac{2}{\tilde{N}}}$$
Noise distributions

2 d.o.f.
4 d.o.f.
6 d.o.f.
8 d.o.f.
10 d.o.f.
Probability gain of the filter

2 d.o.f.
4 d.o.f.
6 d.o.f.
8 d.o.f.
10 d.o.f.
Detecting periodic sources

The problem with this procedure is that the computing cost of the coherent step is intolerably high (because of the higher resolution enhancement and of the spectral filtering). So we need a different hierarchical method.

The main point is that a periodic source is permanent. So one can check the “reality” of a source candidate with the same antenna (or with another of comparable sensitivity) just doing other observations.

So we search for “coincidences” between candidates in different periods.
New hierarchical method

- The analysis is performed by sub-periods (e.g. 4 months)
- For each sub-period the analysis consists only in the first incoherent step, then the candidates (e.g. $10^9$) are archived
- When one has the candidates for at least two periods, one takes the coincidences and does the coherent step (and the following) on the coincident candidates
- The cost of the follow-up is drastically reduced (of the order of $10^6$ less) and so the spectral filtering and the bigger resolution enhancement are no more a problem.