Parameterized post-Newtonian theory of reference frames, multipolar expansions, and equations of motion in the N-body problem*

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Motivation

- The IAU 2000 resolutions recommend to use ***relativistic*** reference frames for processing astronomical observations

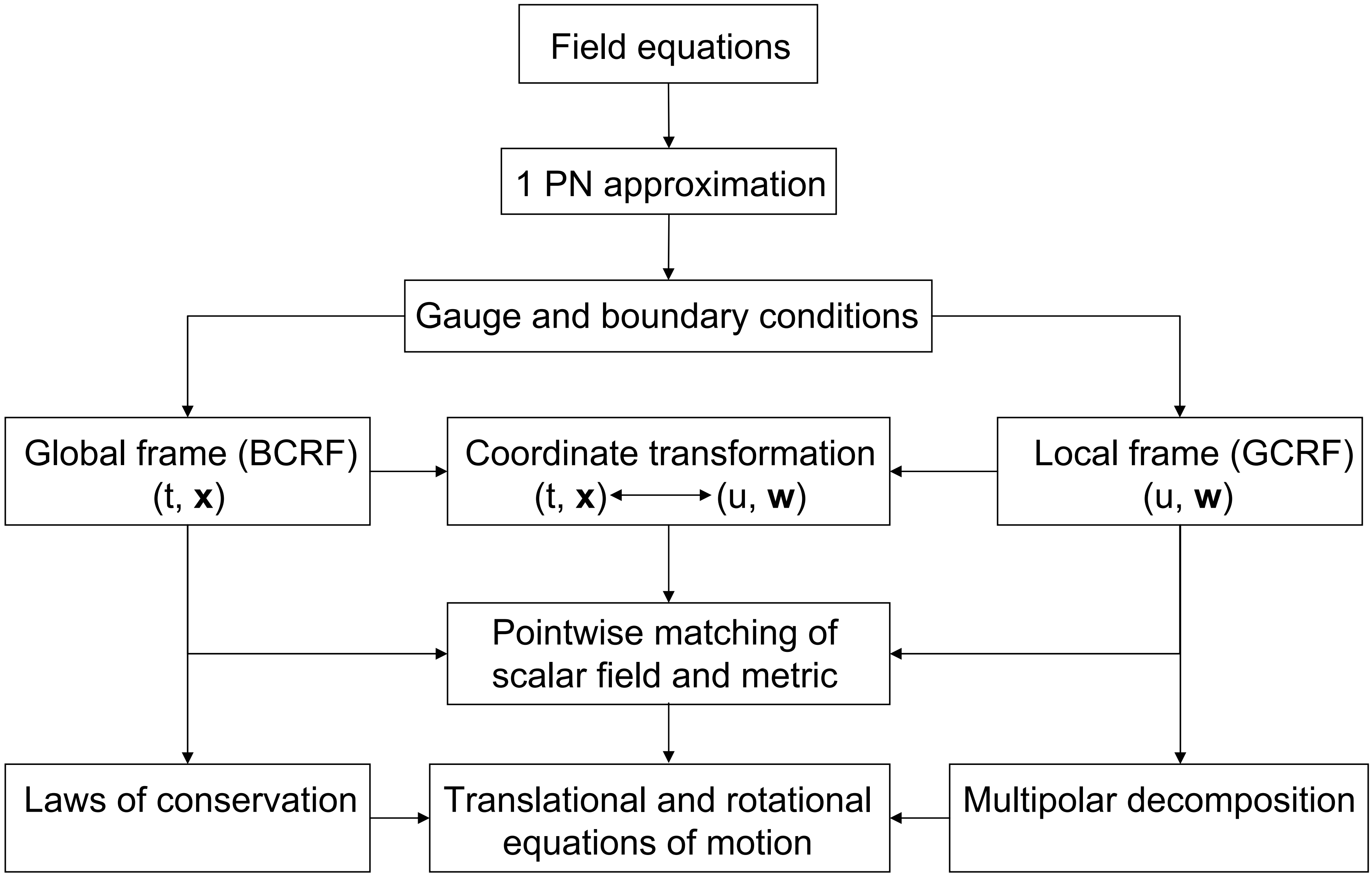
- Alternative theories of gravity provide a more general approach for an analysis of any type of gravitational experiments

- Classical PPN formalism has to be modified significantly in the sense of introduction of multiple reference frames and splitting effects due to the fields different in nature

- Scalar-tensor theory is a good start for modernization of PPN formalism

- Experiments involving determination of parameters γ and β should use an adequate framework of scalar-tensor theories

Scalar-tensor theory



Scalar-tensor theory: Action and field equations

Field equations in the scalar-tensor theory are derived from the action

$$S = \frac{1}{16\pi c^4}\int\left(\phi R - \theta(\phi)\frac{\phi^{,\alpha}\phi_{,\alpha}}{\phi} - \frac{16\pi}{c^4}\mathcal{L}(\Psi, g)\right)\sqrt{-g}\, d^4x$$

Then the equations of gravitational and scalar fields are

$$R_{\mu\nu} = \frac{8\pi}{\phi c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \theta(\phi)\frac{\phi_{,\mu}\phi_{,\nu}}{\phi^2} + \frac{1}{\phi}\left(\phi_{;\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box_g\phi\right)$$

$$\Box_g\phi = \frac{1}{3+2\theta(\phi)}\left(\frac{8\pi}{c^2}T - \phi_{,\alpha}\,\phi^{,\alpha}\,\frac{d\theta}{d\phi}\right)$$

The following post-Newtonian parameters can be introduced:

$$\gamma = \frac{\omega+1}{\omega+2}$$

$$\beta = 1 + \frac{\omega'}{(2\omega+3)(2\omega+4)^2}$$

where ω and ω' are the background values of θ and $d\theta/d\phi$.

Energy-momentum tensor of self-gravitating system of N bodies

Matter comprising the bodies is treated as a fluid with anisotropic tensor of stresses. Density of matter is compactly supported and all bodies are arbitrarily shaped.

Energy-momentum tensor is taken as

$$T^{\mu\nu} = \rho\left(c^2 + \Pi\right) u^\mu u^\nu + \pi^{\mu\nu}$$

where Π is internal energy density,

$\pi^{\mu\nu}$ – tensor of stresses,

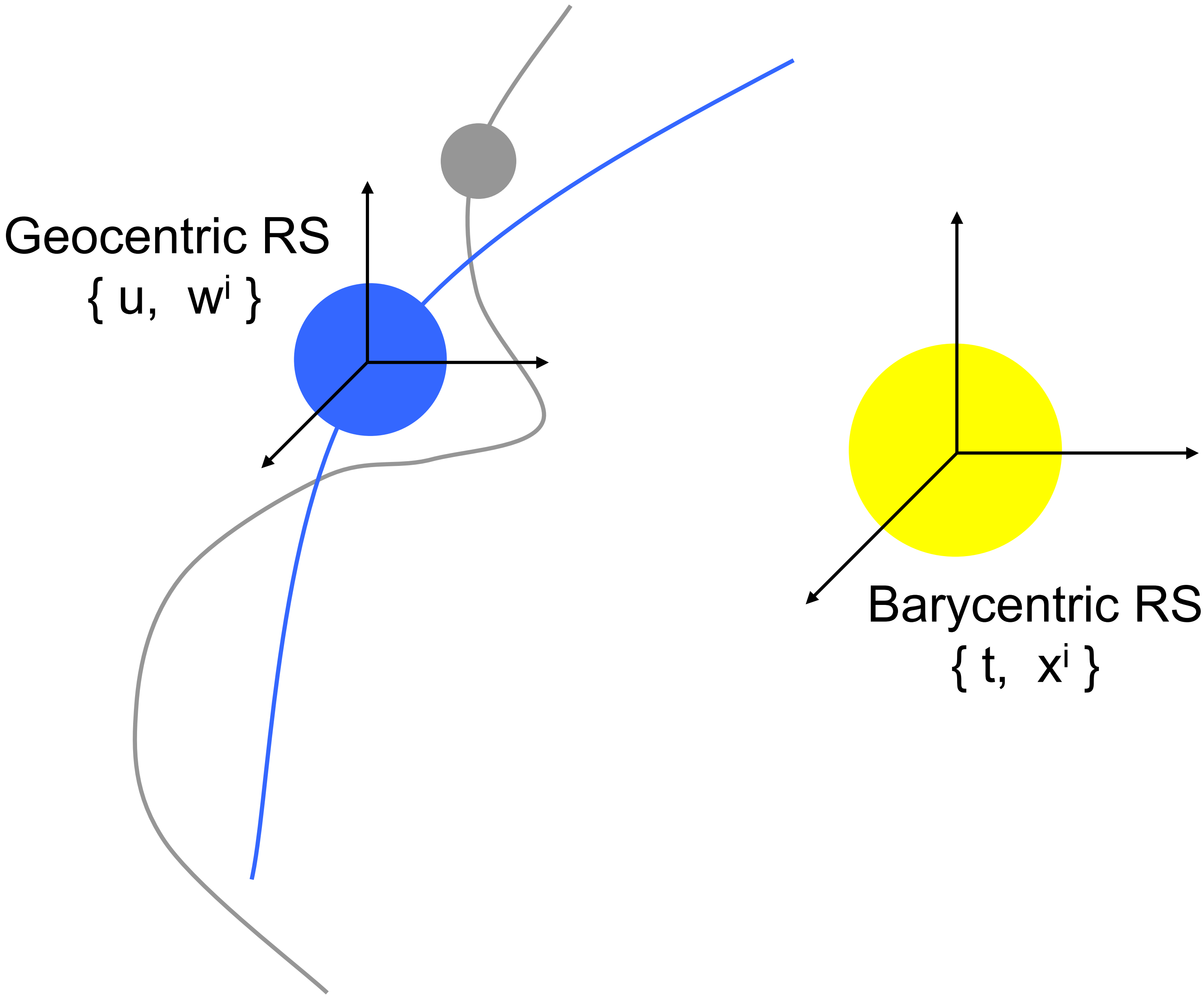
u^μ – 4-velocity of matter.

Invariant density $\rho^* = \sqrt{-g}u^0\rho$ satisfies the exact Newtonian-like equation of continuity

$$\rho^*_{,0} + (\rho^* v^i)_{,i} = 0$$

Energy-momentum tensor is conserved in the covariant sense:

$$T^{\mu\nu}{}_{;\nu} = 0$$


Geocentric RS
{ u, w^i }
Barycentric RS
{ t, x^i }

Global and local frames: Solution of field equations

In the global (inertial) frame the scalar field and the metric tensor are of the form

$$\varphi(t,\boldsymbol{x}) = \frac{1-\gamma}{c^2}U(t,\boldsymbol{x})$$

$$g_{00}(t,\boldsymbol{x}) = -1+\frac{2}{c^2}U(t,\boldsymbol{x})-\frac{2\beta}{c^4}U^2(t,\boldsymbol{x})+\frac{1}{c^4}(\text{ other PN terms })$$

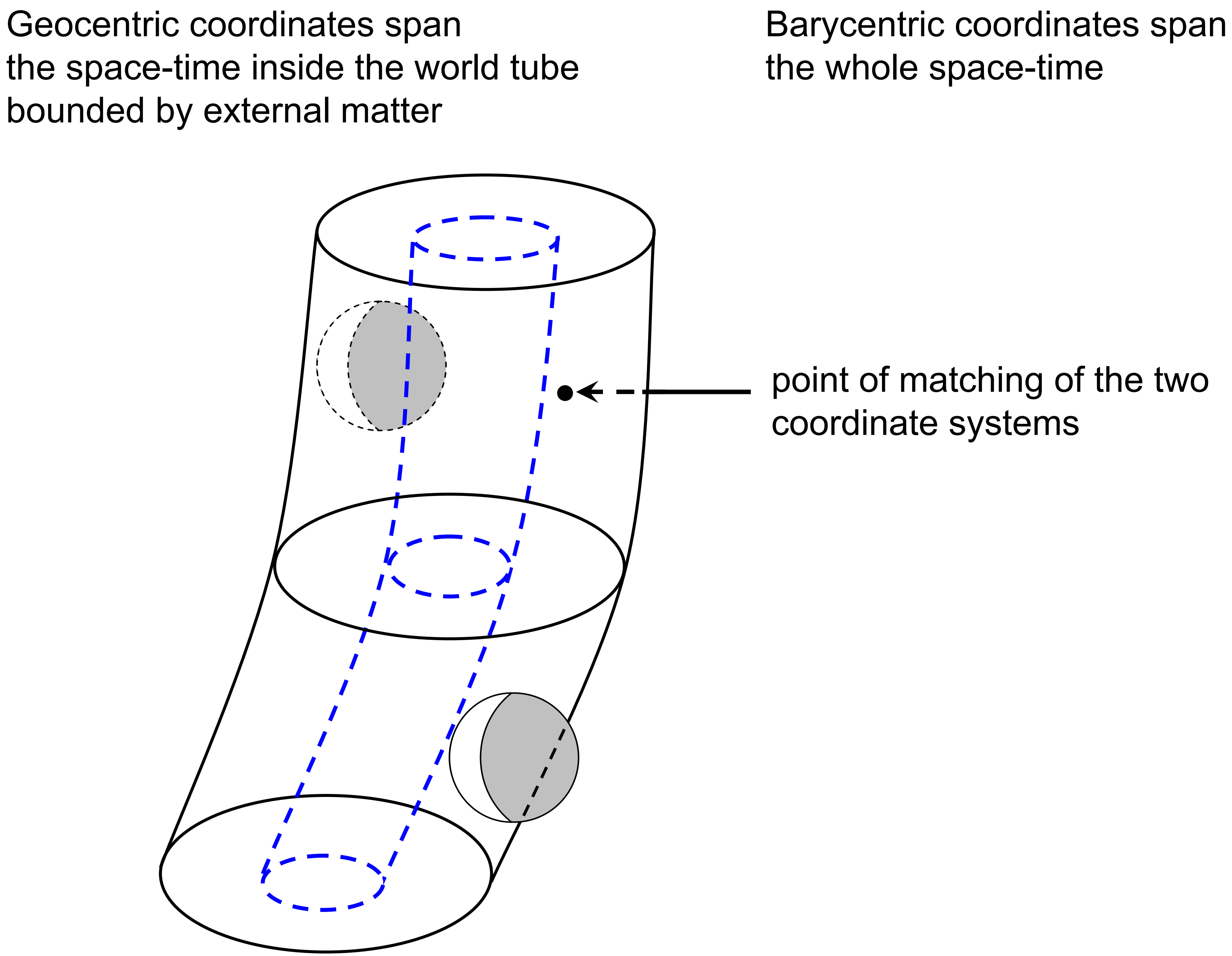
Here $U(t,\boldsymbol{x})$ is the Newtonian potential

$$U(t,\boldsymbol{x}) = \int\frac{\rho^*(t,\boldsymbol{x}')d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

In the local frame the solutions have internal, external, and coupled parts:

$$\hat{\varphi}(u,\boldsymbol{w}) = \frac{1-\gamma}{c^2}\left(\hat{U}_B(u,\boldsymbol{w})+\sum_{l=0}^{\infty}\frac{1}{l!}P_{<L>}(u)w^L\right)$$

$$\hat{g}_{00}(u,\boldsymbol{w}) = -1+\frac{2}{c^2}\left(\hat{U}_B(u,\boldsymbol{w})+Q_i(u)w^i+\sum_{l=2}^{\infty}\frac{1}{l!}Q_{<L>}(u)w^L\right)+\frac{1}{c^4}(\text{ PN terms })$$



Geocentric coordinates span
the space-time inside the world tube
bounded by external matter
Barycentric coordinates span
the whole space-time
point of matching of the two
coordinate systems

Matching: Idea and results

Principle of covariance tells how to find the correspondence between the scalar field and the metric written in different coordinates. In the region of construction of the local frame

$$\begin{aligned} \varphi(t,\boldsymbol{x}) &= \hat{\varphi}(u,\boldsymbol{w}) \\ g_{\mu\nu}(t,\boldsymbol{x}) &= \hat{g}_{\alpha\beta}(u,\boldsymbol{w})\frac{\partial w^\alpha}{\partial x^\mu}\frac{\partial w^\beta}{\partial x^\nu} \end{aligned}$$

Matching of Newtonian and post-Newtonian terms gives the expressions of gravitoelectric and gravitomagnetic moments in terms of external potentials and their derivatives taken at the origin of the local frame:

$$P(t) = \bar{U}\,|_{(\mathbf{w}=\mathbf{0})} + O(c^{-2})$$

$$P_i(t) = \bar{U}_{,i}\,|_{(\mathbf{w}=\mathbf{0})} + O(c^{-2})$$

$$P_{<L>}(t) = Q_{<L>}(t) + O(c^{-2}) = \bar{U}_{,L}\,|_{(\mathbf{w}=\mathbf{0})} + O(c^{-2}) \qquad \forall\, l \geq 2$$

Matching: Results (continued)

Matching of post-Newtonian terms also gives the equations of motion of the origin of the local frame with respect to the global coordinates:

$$a_B^i = \bar{U}_{,i}|_{(\mathbf{w}=\mathbf{0})} - Q_i + \frac{1}{c^2}(\text{ PN terms })$$

Inertial acceleration Q_i can be calculated if the center of mass of the body under consideration is fixed at the origin of the local frame.

Matching procedure also allows to fix the coordinate transformations between the frames completely. For instance, in the Newtonian limit the time transforms as

$$u = t - \frac{1}{c^2}(\mathcal{A} + \boldsymbol{v}_B \cdot \boldsymbol{w}) + \frac{1}{c^4}(\text{ PN terms })$$

where

$$\dot{\mathcal{A}} = \frac{1}{2}(\boldsymbol{v}_B)^2 + \bar{U}|_{(\mathbf{w}=\mathbf{0})}$$

Global conservation laws

Representing the field equations in the form

$$\Theta^{\mu\nu} \equiv (-g)\phi\Big[c^2T^{\mu\nu}+t^{\mu\nu}\Big] = \frac{c^4}{16\pi}\Big[(-g)\phi^2(g^{\mu\nu}g^{\alpha\beta}-g^{\mu\alpha}g^{\nu\beta})\Big]_{,\alpha\beta}$$

the conservation laws can be obtained:

$$\Theta^{\mu\nu}{}_{,\nu} = \Big[(-g)\phi(c^2T^{\mu\nu}+t^{\mu\nu})\Big]_{,\nu} = 0$$

Therefore, the conserved mass $\mathbb{M}$, the linear momentum $\mathbb{P}^i$, and the spin $\mathbb{S}^i$ of an isolated N-body system can be defined as

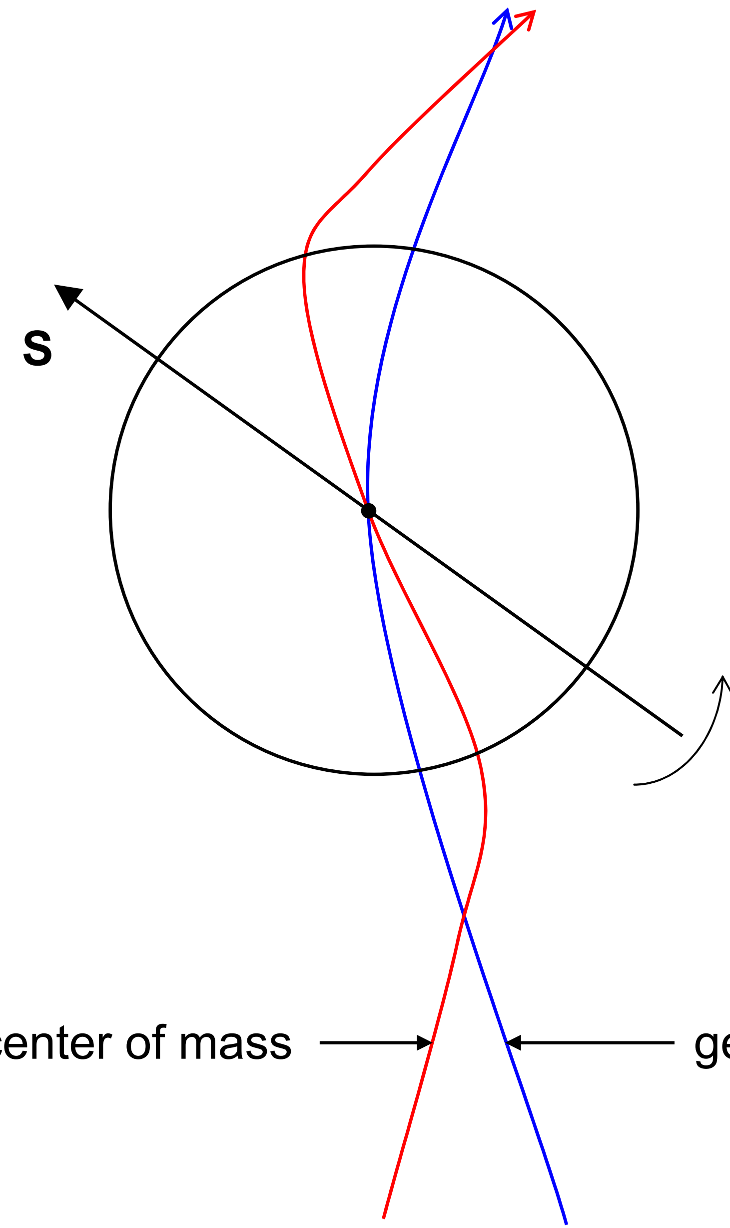
$$\mathbb{M} = \frac{1}{c^2}\int_{\mathcal{D}}\Theta^{00}\,d^3x = \tilde{I}$$

$$\mathbb{P}^i = \frac{1}{c}\int_{\mathcal{D}}\Theta^{0i}\,d^3x = \frac{d}{dt}\tilde{I}^i$$

$$\mathbb{S}^i = \frac{1}{c}\int_{\mathcal{D}}\varepsilon^i{}_{jk}x^j\Theta^{0k}\,d^3x$$

Here $\tilde{I}$ is the mass and $\tilde{I}^i$ is the dipole moment of the ***conformal*** metric $\tilde{g}_{\mu\nu} \equiv (1+\varphi)g_{\mu\nu}$

Center of mass of a finite size massive body doesn’t move along a geodesic. This effect exists even in the Newtonian approximation.



The center of mass moves with the acceleration Q_i with respect to the geodesic passing through its initial position

For the Earth this acceleration amounts to $3 \cdot 10^{-11}$ m/s^2

The geodetic acceleration is about $6 \cdot 10^{-3}$ m/s^2

so the effect is of order $5 \cdot 10^{-9}$

world line of the center of mass

geodesic in the global RS

Center of mass and translational equations of motion

Since the ***conformal*** dipole moment $\tilde{I}^i$ of an isolated body is conserved, it's natural to define the center of mass of the body under consideration by conditions

$$\tilde{I}^i(u) = \frac{d}{du}\tilde{I}^i(u) = \frac{d^2}{du^2}\tilde{I}^i(u) = 0 \qquad \forall\ u$$

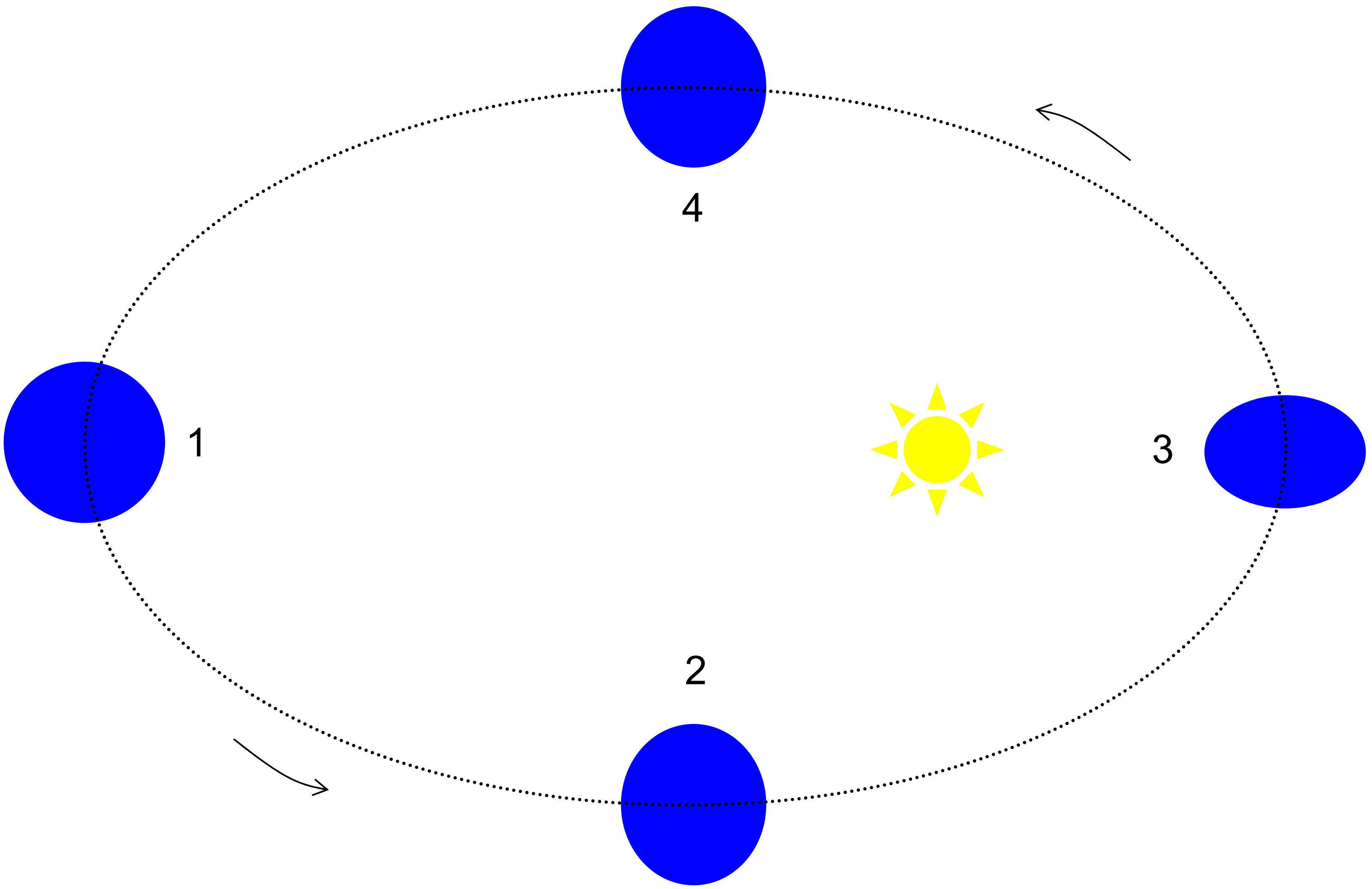
If they are satisfied, the origin of the local frame is fixed at the center of mass. These conditions lead to the expressions for yet unknown functions Q_i:

$$\tilde{\mathcal{M}}_B Q_i = -\sum_{l=1}^{\infty} \frac{1}{l!} Q_{<iL>}(u)\mathcal{I}^L(u) + \frac{1}{c^2}(\text{ PN terms })$$

This completes the derivation of translational equations of motion – the center of mass of body B has an acceleration given in the global coordinates by

$$a_B^i = \bar{U}_{,i}|_{(\mathbf{w}=0)} - Q_i + \frac{1}{c^2}(\text{ PN terms })$$

Spherical symmetry of a moving body is not well-defined in the global frame:



Case of spherically symmetric and rigidly rotating bodies

Spherical symmetry in the local frame implies that

$$\rho^*(u, \boldsymbol{w}) = \rho^*(r) , \qquad \Pi(u, \boldsymbol{w}) = \Pi(r) , \qquad \pi^{ij}(u, \boldsymbol{w}) = \delta^{ij} p(r)$$

Also for any $l \geq 1$ one has

$$\int_{V_B} f(r) w^{<i_1 i_2 \ldots i_l>} d^3 w \; = \; 0$$

Therefore, the multipolar expansion of the Newtonian potential of the body in the local coordinates contains a monopole term only:

$$U_B(u, \boldsymbol{w}) = G \int_{V_B} \frac{\rho^*(u, \boldsymbol{w}') d^3 w'}{|\boldsymbol{w} - \boldsymbol{w}'|} = \frac{G\mathcal{M}_{*B}}{r}$$

Note that such expression is valid in the local frame only. There is no symmetry in the global frame and the potential has some additional multipoles: $[R^i_B = x^i - x^i_B]$

$$\mathbb{I}^L_B \; = \; \int_{V_B} \rho^*(t, \boldsymbol{x}) R^{i_1}_B R^{i_2}_B \ldots R^{i_l}_B d^3 x$$

$$\mathbb{I}^i_B \; = \; \frac{1}{3c^2} \mathcal{I}^{(2)}_B \left(\varepsilon_{ijk} v^j_B \Omega^k_B + \frac{1}{2} a^i_B \right) + O\left(c^{-4}\right)$$

$$\mathbb{I}^{<ij>}_B \; = \; -\frac{1}{3c^2} \mathcal{I}^{(2)}_B v^{<i}_B v^{j>}_B + O\left(c^{-4}\right)$$

$$\mathbb{I}^{<L>}_B \; = \; O\left(c^{-4}\right) , \qquad (l \geq 3)$$

Translational motion

Define the general relativistic mass M_B and Nordtvedt gravitational mass $\mathfrak{M}_B$:

$$\mathrm{M}_B = \int_{V_B} \rho^* \left[1 + \frac{1}{c^2}\left(\frac{1}{2}\nu^2 + \Pi - \frac{1}{2}\hat{U}^{(B)}\right)\right] d^3w$$

$$\mathfrak{M}_B = \mathrm{M}_B - \frac{\eta}{2c^2}\int_{V_B} \rho^* \hat{U}_B d^3w$$

Then the acceleration of the center of mass of the body B is given by

$$\mathrm{M}_B a_B^i = F_N^i + \frac{1}{c^2}\left[F_{EIH}^i + + F_{\mathcal{S}}^i + F_{\mathcal{IGR}}^i + \delta F_{\mathcal{IGR}}^i\right] + O(c^{-4})$$

The first term represents the Newtonian force

$$F_N^i = \sum_{C \neq B} \frac{G \mathfrak{M}_B \mathfrak{M}_C R_{BC}^i}{R_{BC}^3}$$

$$
\begin{aligned}
F^{i}_{EIH} &= \sum_{C\neq B}\frac{GM_B M_C R^i_{BC}}{R^3_{BC}}\Bigg\{\gamma v_B^2 - 2(1+\gamma)(\boldsymbol{v}_B\cdot\boldsymbol{v}_C) + (1+\gamma)v_C^2 \\
&- \frac{3}{2}\left(\frac{\boldsymbol{R}_{BC}\cdot\boldsymbol{v}_C}{R_{BC}}\right)^2 - (1+2\gamma+2\beta)\frac{GM_B}{R_{BC}} - 2(\gamma+\beta)\frac{GM_C}{R_{BC}} \\
&+ \sum_{D\neq B,C}\left[(1-2\beta)\frac{GM_D}{R_{CD}} - 2(\gamma+\beta)\frac{GM_D}{R_{BD}} + \frac{GM_D(\boldsymbol{R}_{BC}\cdot\boldsymbol{R}_{CD})}{2R^3_{CD}}\right]\Bigg\} \\
&+ \sum_{C\neq B}\Bigg\{\frac{GM_B M_C(v^i_C - v^i_B)}{R^3_{BC}}\left[2(1+\gamma)(\boldsymbol{v}_B\cdot\boldsymbol{R}_{BC}) - (1+2\gamma)(\boldsymbol{v}_C\cdot\boldsymbol{R}_{BC})\right] \\
&+ \frac{3+4\gamma}{2}\frac{GM_B M_C}{R_{BC}}\sum_{D\neq B,C}\frac{GM_D R^i_{CD}}{R^3_{CD}}\Bigg\} \\
F^i_S &= G\sum_{C\neq B}\Bigg\{\frac{M_C\mathcal{S}^p_B(v^k_C - v^k_B)}{2R^5_{BC}}\Big[3(1+\gamma)\left(\varepsilon_{ikq}R^{<pq>}_{BC} - \varepsilon_{kpq}R^{<iq>}_{BC}\right) \\
&+ (1-\gamma)\varepsilon_{ipq}R^{<kq>}_{BC}\Big] + 3(1+\gamma)\frac{M_B\mathcal{S}^p_C(v^k_C - v^k_B)}{R^5_{BC}}\left[\varepsilon_{ipq}R^{<kq>}_{BC} - \varepsilon_{kpq}R^{<iq>}_{BC}\right] \\
&- \frac{15(1+\gamma)}{2}\frac{\mathcal{S}^j_B\mathcal{S}^k_C R^{<ijk>}_{BC}}{R^7_{BC}} - \left(\gamma+\frac{1}{2}\right)\frac{R^{<ijk>}_{BC}}{R^7_{BC}}\left[M_B\mathcal{I}^{(4)}_C\Omega^j_C\Omega^k_C + M_C\mathcal{I}^{(4)}_B\Omega^j_B\Omega^k_B\right]\Bigg\} \\
F^i_{IGR} &= -G^2\sum_{C\neq B}\sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}\Bigg[(-1)^l M_B\mathcal{I}^{(2l)}_C\frac{R^{<iL>}_{BC}}{R^{2l+3}_{BC}}\sum_{D\neq C}\frac{M_D R^{<L>}_{CD}}{R^{2l+1}_{CD}} \\
&+ M_C\mathcal{I}^{(2l)}_B\frac{R^{<L>}_{BC}}{R^{2l+1}_{BC}}\sum_{D\neq B}\frac{M_D R^{<iL>}_{BD}}{R^{2l+3}_{BD}}\Bigg] \\
\delta F^i_{IGR} &= 2(1-\beta)G^2\sum_{C\neq B}\Bigg\{M_C\mathcal{I}^{(2)}_B\frac{R^k_{BC}}{R^3_{BC}}\sum_{D\neq B}\frac{M_D R^{<ik>}_{BD}}{R^5_{BD}} \\
&+ M_B\mathcal{I}^{(2)}_C\frac{R^{<ik>}_{BC}}{R^5_{BC}}\sum_{D\neq C}\frac{M_D R^k_{CD}}{R^3_{CD}} + \sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}\Bigg[(-1)^l M_B\mathcal{I}^{(2l)}_C\frac{R^{<iL>}_{BC}}{R^{2l+3}_{BC}}\sum_{D\neq C}\frac{M_D R^{<L>}_{CD}}{R^{2l+1}_{CD}} \\
&+ M_C\mathcal{I}^{(2l)}_B\frac{R^{<L>}_{BC}}{R^{2l+1}_{BC}}\sum_{D\neq B}\frac{M_D R^{<iL>}_{BD}}{R^{2l+3}_{BD}}\Bigg]\Bigg\}
\end{aligned}
$$

Rotational motion

The rotational equations of motion for the body's spin are

$$\frac{d\mathcal{S}^i}{du} = \frac{1}{c^2}\left(\frac{2\gamma+1}{15}\varepsilon_{ijk}Q_{jn}\Omega_B^k\Omega_B^n\mathcal{I}_B^{(4)} - 2(\gamma+1)\varepsilon_{ijk}\mathcal{S}^j C_k\right) + O(c^{-4})$$

where C_k is the angular velocity of rotation of the local coordinate frame with respect to that which axes are subject to the Fermi-Walker transport.