# Upper limits on burst sources of gravitational waves

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We present statistical methods of determining upper limit event rates on burst sources using gravitational wave detectors. By assuming that burst events are Poisson distributed in time, we devise two experiments to estimate the event rate. The first is a generalization of the loudest event statistic used by Allen *et al*. [B. Allen *et al*., Phys. Rev. Letters **83**, 1498 (1999)] to determine an upper limit on the number of binary neutron star inspirals in the Galaxy using data taken in 1994 using the LIGO 40-meter prototype interferometer at Caltech. If the probability distribution of background events can be determined, this method could be used to improve the event rate limit. However, in the absence of accurate knowledge of the distribution in signal to noise ratio of the false alarms, we show that the loudest event statistic determines a robust upper bound. The second method assumes that a detection threshold is set in advance of performing the data analysis, and that the number of events with signal to noise exceeding this threshold is $n$. If the threshold has been chosen so that the probability of a false event is low, then this method determines a lower limit than the loudest event statistic. Nevertheless, we argue that the loudest event statistic provides a robust estimator, and, as such, is preferred when the distribution of signal to noise due to background events is not well understood.

## I. INTRODUCTION

Of the sources of gravitational waves that theorists can imagine, inspiralling binary neutron stars, or black holes, are likely to provide the strongest gravitational wave signals in the band of earth based detectors. Interferometric detectors will take data with a strain sensitivity of around $10^{-XX}$–the gravitational wave strain due to an optimally oriented binary neutron star inspiral located directly above the detector at a distance of XX Mpc. Given the uncertainties in event rate estimates for these sources, some of the first experiments using the new detectors will set upper limits on the rates directly from the gravitational wave data.

In a recent Letter, Allen *et al*. [1] described the analysis of approximately 45 hours of data taken with the LIGO 40-meter prototype interferometer in November 1994. The detector was a prototype; correspondingly, the analysis was a prototype, intended principally to demonstrate an analysis pipeline aimed at determining an upper limit on the galactic neutron star binary inspiral rate. The data was processed by an analysis pipeline, which identified candidate galactic compact binary inspiral events. That pipeline was characterized by two outputs for every segment of data filtered: the S/N $\rho$, corresponding to the maximum signal amplitude from a bank of filters, and another statistic $\chi^2$ which measures the time-frequency distribution of S/N for a given event relative to that expected from a neutron star inspiral event. Large values of $\chi^2$ indicate that the S/N is not accumulated in the manner predicted for binary inspiral signals. A threshold $\chi^2_*$ was used such that less than 10% of real signals would give $\chi^2 > \chi^2_*$ in the presence of stationary, Gaussian noise.

To determine the upper limit, Allen at al. [1] focused on the single event with the greatest signal-to-noise (S/N) ratio that passed all of the cuts in the analysis pipeline, *i.e.*, the loudest event. Since events with greater S/N are generally closer, and therefore rarer, than fainter events. Thus, in some fixed observation time, the probability that the loudest event has a particular amplitude is related to the event rate. Thus, the algorithm underlying the analysis is:

- Analyze the data through the data analysis pipeline,
- Set $\rho_{\max}$ equal to the amplitude of the loudest detected event, *or* if no events are detected, set $\rho_{\max} = 0$,
- Determine the efficiency with which the pipeline can detect binary neutron star inspiral events by Monte Carlo simulation.
- Compute an upper limit based on the number $\rho_{\max}$ and the efficiency of the pipeline at that S/N ratio.

In this paper, we present statistical analyses which determine event rate (upper) limits for signals which are Poisson distributed in time, in the presence of an arbitrary noise background. In the limit that the loudest event is assumed to be a real signal, and the probability of a background event having S/N greater than or equal to that of the loudest event, this reproduces the limit presented in Ref. [1]. We use our formula to demonstrate the robust nature of the limit determined by Allen *et al.*. This indicates that the new formula can be used to provide an improved upper limit *if* the background can be measured.

An alternative method to determine an upper limit event rate using data from the kilometer scale interferometers might include the imposition of a threshold(s) in S/N (and $\chi^2$). We derive the probability distribution for the event rate of real signals given the threshold and the number of events that are detected above the threshold. We compare this event rate limit, for several different outcomes of the experiment, with that obtained using the loudest event statistic.

In section II, we introduce the notation and terminology that is used throughout the paper. We calculate, from first principles, the likelihood that the loudest event has magnitude $\rho_{\max}$ in section III. This result has broad applicability in other contexts. Specializing to the assumptions made in [1], we arrive at the upper limit given there, and provide quantitative evidence of the robust nature of the limit. In section IV, we derive an expression for the likelihood when using threshold detection in the data analysis pipeline. Finally, we present a critical assessment of the relative merits of each method.

## II. NOTATION AND TERMINOLOGY

A data analysis pipeline process identifies *events.* Each identified event is characterized by a single number, its S/N $\rho$. We assume that discrete events are independent and are either *background* or *foreground* events. A background event is a *false alarm:* the signal being sought is not, in fact, present in the data stream at the time of the event. Background events may be caused by instrumental or environmental noise, or by "nuisance" signals (*i.e.*, signals from other sources not being sought). Events corresponding to an actual signal are foreground events, and correspond to a *detection.* Not all signals give rise to foreground events: the analysis pipeline detects only some fraction of the signals. It is a bound on the rate of signals, which give rise to foreground events, that we wish to determine from the observations.

We assume that signal events are Poisson distributed with rate $R_0$, and that the analysis pipeline detects infinitesimally weak signal events with an efficiency $\epsilon$ in the absence of background events. Thus, the number of (Poisson distributed) foreground events in an observation of duration $T$ is

$$P_0(N|\mu_0\epsilon) = \frac{(\mu_0\epsilon)^N}{N!}e^{-\mu_0\epsilon} \tag{1a}$$

where

$$\mu_0 = R_0 T. \tag{1b}$$

Like all events, foreground events are each associated with a S/N $\rho$ which characterizes their amplitude, and which depends on the analysis pipeline.[1] Denote the probability density that a foreground event has S/N $\rho$ by $p_0(\rho)$:

$$p_0(\rho) = \begin{pmatrix} \text{probability that foreground} \\ \text{event has amplitude } \rho \end{pmatrix}. \tag{1c}$$

Finally, denote the probability that a foreground event has S/N less than or equal to $\rho$ by $c_0(\rho)$:

$$c_0(\rho) = \int_0^\rho dx\, p_0(x) \tag{1d}$$

## III. LOUDEST EVENT STATISTICS

In this section, we derive the posterior probability distribution for the event rate $\mu_0$ given a detection (either background or signal) with S/N $\rho_{\text{max}}$, i.e. $P[\mu_0|\rho = \rho_{\max}]$. The probability that all foreground events, regardless of their number, have S/N less than or equal to $\rho$ is

$$P_0(\leq \rho|\mu_0) = P_0(0|\mu_0) + \sum_{N=1}^{\infty} P_0(N|\mu_0)c_0(\rho)^N \tag{2}$$

$$= \exp[-\epsilon\mu_0(1 - c_0(\rho))] \,. \tag{3}$$

No assumptions are made about the distribution of signal to noise ratio for background events. We simply denote the (cumulative) probability that all background events, regardless

[1] The matched filtering S/N depends on the signal being filtered for, and on the power spectral estimate used in that filter.

of their number, have S/N less than or equal to $\rho$ by $F_b(\rho)$. Finally, the probability of that an event, either foreground or background, has S/N less than or equal to $\rho$ is

$$P(\leq \rho|\mu_0, b) = P_0(\leq \rho|\mu_0)F_b(\rho) \ . \tag{4}$$

We note an important point about the cumulative probability (3). Since the limit as $\rho \to 0$ is non-vanishing, i.e.

$$\lim_{\rho\to 0}\left[P_0(\leq \rho|\mu_0)\right] = e^{-\epsilon\mu_0} \ , \tag{5}$$

the associated probability density function is distributional at $\rho = 0$. In particular, the probability that an event has S/N $\rho$ is

$$\begin{aligned} p(\rho|\mu_0, b) &= \frac{d}{d\rho}P(\leq \rho|\mu_0, b) \\ &= F_b(\rho)\left[\epsilon\mu_0 p_0(\rho)e^{-\epsilon\mu_0(1-c_0(\rho))} + e^{-\epsilon\mu_0}\Delta(\rho)\right] + e^{-\epsilon\mu_0(1-c_0(\rho))}f_b(\rho) \ , \end{aligned} \tag{6}$$

where $f_b(\rho) = dF_b(\rho)/d\rho$, and $\Delta(\rho)$ is a distribution[2] such that $\Delta(\rho) = 0$ when $\rho \neq 0$, and

$$\int_0^\rho \Delta(x)dx = 1 \ . \tag{7}$$

Using Bayes law, we can write the probability (meaning degree of belief) the $\mu_0$ takes on a particular value as

$$P(\mu_0|\rho = \rho_{\max}) = \frac{p(\rho|\mu_0, b)p(\mu_0)}{p(\rho|b)} \tag{8}$$

where $p(\mu_0)$ is the *a priori* probability density for $\mu_0$, and $p(\rho_{\max}|b)$ is the probability density for $\rho_{\max}$ no matter what the event rate $\mu_0$:

$$p(\rho_{\max}|b) = \int_0^\infty d\mu_0 p(\rho_{\max}|\mu_0, b)p(\mu_0) \ . \tag{9}$$

Assuming an improper uniform probability density $p(\mu_0)$, we can evaluate these expressions explicitly to get

$$P(\mu_0|\rho = \rho_{\max}) = \mu_0[\epsilon(1 - c_0(\rho))]^2 e^{-\epsilon\mu_0(1-c_0(\rho))}\mathcal{N} \tag{10}$$

where

$$\mathcal{N} = \frac{F_b(\rho_{\max})p_0(\rho_{\max}) + f_b(\rho_{\max})/(\mu_0\epsilon)}{F_b(\rho_{\max})p_0(\rho_{\max}) + f_b(\rho_{\max})(1 - c_0(\rho_{\max}))} \ . \tag{11}$$

An upper limit on the event rate is determined by solving the equation

$$\begin{aligned} p &= \int_0^{\mu_p} d\mu_0 P(\mu_0|\rho = \rho_{\max}) \\ &= 1 - \frac{e^{-\epsilon\mu_0(1-c_0(\rho))}[\epsilon(1 - c_0(\rho_{\max}))(F_b\mu_p + f_b/p_0(\rho_{\max})) + F_b]}{F_b + \epsilon f_b(1 - c_0(\rho_{\max}))/p_0(\rho_{\max})} \end{aligned} \tag{12}$$

for $\mu_p$, given the desired confidence $p$, the efficiency $\epsilon$, and the probabilities $c_0(\rho_{\max})$, $p_0(\rho_{\max})$, $F_b(\rho_{\max})$ and $f_b(\rho_{\max})$.

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[2]Since the integral in Eq. (7) is from zero to $\rho$, it is technically incorrect to call this a $\delta$-function. Nevertheless, a physicist can understand it in those terms

### A. Weak upper limit calculation

In some circumstances, it is extremely difficult (if not impossible) to determine the false alarm probability $F_b(\rho)$ with any degree of confidence. Such a circumstance was encountered by Allen et al. [1], who made some assumptions to obtain a weak, *but* robust, upper limit on the event rate $\mu_p$.

We enumerate the assumptions as: (i) the loudest detected event from the data analysis pipeline is a signal, and (ii) the probability $F_b(\rho_{\rm max}) = 0$. Then, Eq. (12) reduces to

$$p = 1 - e^{-\epsilon\mu_p(1-c_0(\rho_{\rm max}))}[1 + \epsilon\mu_p(1 - c_0(\rho_{\rm max}))] \ . \quad (13)$$

This implicit equation can be solved for $\epsilon\mu_p(1 - c_0(\rho_{\rm max}))$ given $p$; the results are shown in Fig. 1. In particular, the $90\%$ confidence limit on the event rate is

$$\mu_{90\%} = \frac{3.890}{\epsilon(1 - c_0(\rho_{\rm max}))} \ . \quad (14)$$

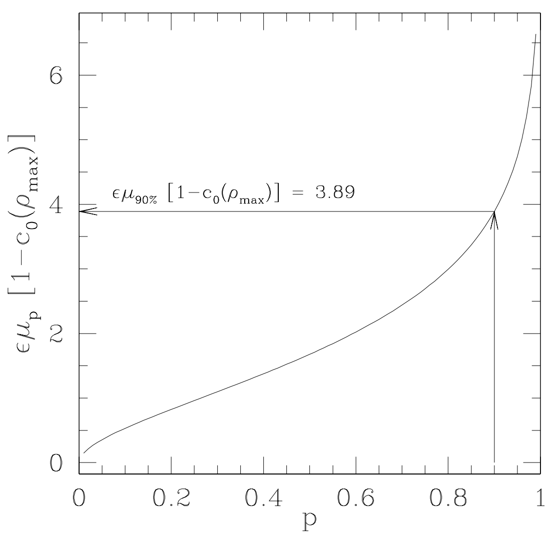


FIG. 1. The value of $\mu_p$ as a function of the confidence $p$ computed with Eq. (13). The $90\%$ confidence limit is indicated.

As shown in Ref. [1], the combination $\epsilon(1 - c_0(\rho_{\rm max}))$ can be determined by Monte-Carlo injection of simulated signals (from the population to be constrained) into the data, and reanalyzing it. Using the data analysis pipeline described in Ref. [1], the S/N $\rho_{\rm max}$ of the loudest event was found to be $\rho_{\rm max} = 8.34$ and $\epsilon(1 - c_0(\rho_{\rm max})) \simeq 0.67$ for neutron star binary inspiral signals from a Galactic population; this gives the rate limit $\mu_{90} = 11.788$ quoted in Ref. [1].

### B. The weak upper limit is robust

The effects of the background is a major concern when detector noise is not well understood. While one can assume, following Allen *et al.*, that the probability of a background event with S/N greater than or equal to $\rho_{\max}$ is zero, one would like to understand the errors that can arise from this assumption. Equation (12) can be used to quantify these errors by writing $\mu_p = 11.788$, and determining $p$ as a function of $F_b(\rho_{\max})$ and $f_b(\rho_{\max})/\epsilon p_0(\rho_{\max})$. If $p > 0.9$, the we say that the upper limit is robust. That is, with more information about the distribution $F_b(\rho)$, we could improve our limit on the event rate. Figure 2 demonstrates that the limit is robust when $\epsilon(1 - c_0(\rho_{\max})) = 0.67$ and $\mu_{90\%} = 11.788$.

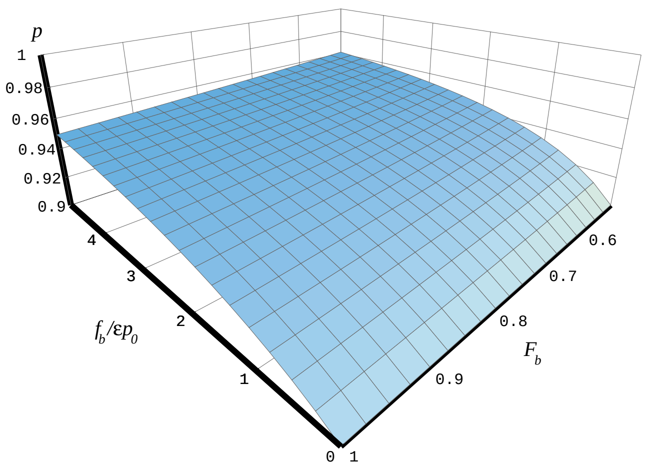


FIG. 2. The probability that $\mu \leq \mu_{90\%} = 11.788$ as a function of $F_b(\rho_{\max})$ and $f_b(\rho_{\max})/\epsilon p_0(\rho_{\max})$. Since this probability is always greater than 0.9, we conclude that the weak limit discussed in Sec. III B is robust.

## IV. THRESHOLD SEARCHES AND UPPER LIMITS

The hope for gravitational wave detection during the next decade is that we can understand the noise in the detectors by using the network of detectors at different locations around the world. A natural question, then, is how to set an upper limit when we have a good understanding of the noise in our detector. In this section, we explore this question in the context of a search for gravitational wave events with $\rho \geq \rho_*$ where $1 - F_b(\rho_*) \ll 1$. The output of the data analysis pipeline is the number of events $n$ which exceed this threshold. Then the probability of interest is

$$p[\mu_0 | n \text{ events with } \rho \geq \rho_*] = \frac{P[n \text{ events with } \rho \geq \rho_* | \mu_0] p[\mu_0]}{P[n \text{ events with } \rho \geq \rho_*]} \,. \tag{15}$$

First, the probability of detecting $n$ events with $\rho \geq \rho_*$ given some event rate $\mu_0$ is

$$P[n \text{ events with } \rho \geq \rho_* | \mu_0] = \sum_{i=0}^{n} \binom{n}{i} P[n \text{ signals with } \rho \geq \rho_* | \mu_0] F_i \tag{16}$$

where

$$F_i = P[i \text{ false alarms with } \rho \geq \rho_*] \ . \tag{17}$$

Note that $F_0 = F_b(\rho_*) \simeq 1$ and $\sum_{i=1}^{\infty} F_i = (1 - F_0) \ll 1$, so that $F_i \ll 1$ for all $i$. Thus, we have the probability that $n$ events with $\rho \geq \rho_*$ are recorded by the data analysis pipeline:

$$P[n \text{ events with } \rho \geq \rho_* | \mu_0] = \sum_{i=0}^{n} \binom{n}{i} F_i \frac{[\mu_0 \epsilon (1 - c_0(\rho_*))]^{n-i}}{(n-i)!} e^{-\mu_0 \epsilon (1 - c_0(\rho_*))} \tag{18}$$

$$\simeq F_0 \frac{[\mu_0 \epsilon (1 - c_0(\rho_*))]^{n}}{n!} e^{-\mu_0 \epsilon (1 - c_0(\rho_*))} \ . \tag{19}$$

Clearly, this approximation should be valid whenever

$$F_i \ll F_0 \left[ \frac{(n-i)!}{n!} \right]^2 [\mu_0 \epsilon (1 - c_0(\rho_*))]^i i! \ . \tag{20}$$

Using the improper uniform prior on $\mu_0$ as in the previous section, we arrive at our result

$$P[\mu < \mu_p | n \text{ events with } \rho \geq \rho_*] \simeq 1 - \frac{\Gamma[n+1, \epsilon(1 - c_0(\rho_*))\mu_p]}{n!} \tag{21}$$

When $n = 0$ events are detected by the data analysis pipeline, the $90\%$ confidence limit on $\mu$ is

$$\mu_{90\%} = \frac{2.3}{\epsilon (1 - c_0(\rho_*))} \tag{22}$$

which would be $40\%$ lower than that obtained using the loudest event statistic *if* $\rho_* = \rho_{\max}$.

As in the case of the loudest event statistic, it is important to ascertain the validity of the approximation that is made in Eq. (19). This can be done by examining Eq. (20) when $\mu_0 = \mu_{90\%}$. Interestingly, the correction terms become more important as the number of detected events increases. In particular, we expect the approximation to be reasonable when $n \leq 2$.

## V. CONCLUSIONS

**I THINK WE HAVE SEVERAL CONCLUSIONS: 1) THE LOUDEST EVENT STATISTIC HAS BEEN DERIVED IN THE PRESENCE OF AN ARBITRARY BACKGROUND, 2) THE UPPER LIMIT DETERMINED USING THE LOUDEST EVENT IS ROBUST, 3) AN UPPER LIMIT ANALYSIS HAS BEEN PERFORMED BASED ON A THRESHOLD DETECTION PIPELINE, 4) THIS LIMIT IS 40% SMALLER THAN THE LOUDEST EVENT LIMIT WHEN NO EVENTS ARE DETECTED, 5) OUR APPROXIMATION IN THE THRESHOLD CASE BREAKS DOWN AS THE NUMBER OF EVENTS INREASES BEYOND $N = 3$. WE SHOULD ALSO ADD A DISCUSSION OF WAYS TO**

**ESTIMATE THE BACKGROUND USING ALBERT'S SUGGESTION, MEASURING IT FOR MULTIPLE DETECTORS, AND MAYBE USING TIME-REVERSED CHIRP FILTERS (ORIGINALLY DUE TO STUART ANDERSON, AND INDEPENDENTLY BY KENT BLACKBURN). IT WOULD BE NICE TO HAVE EACH OF THESE CLEARLY SUGGESTED IN THE LITERATURE. — Patrick.**

We have derived a general and exact expression for the probability that the "loudest" event — *i.e.*, the event of greatest signal-to-noise ratio — drawn with arbitrary efficiency from a population of Poisson distributed signals and in the presence of a Poisson distributed background takes on a particular value.

Improvements in the upper limit (still remaining in the framework of loudest event analyses) can be made by estimating the background event rate. For example, one could assume that the events between S/N of 6 and 7, shown in figure 2 of [1], are due to background and determine a power-law for the expected number of background events at higher S/N, allowing the assumption of no background events to be relaxed. Alternatively, Lazzarini [2] has suggested that a rudimentary estimate of the background rate could be obtained by evaluating the distribution of the detector output samples and, using that distribution together with an appropriate linear filter to reproduce the noise power spectrum, simulate and analyze detector output. Signals present in the original detector output will, by construction, not be present in the simulated output[3]; consequently, only background events emerge at the end of the analysis pipeline when evaluated on the simulated data, allowing the background event rate to be estimated.[4]

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[1] B. Allen *et al.*, Phys. Rev. Letters **83**, 1498 (1999), `gr-qc/9903108`.
[2] A. Lazzarini, Private communication, 1999.

[3] Their power will be, but without the correlations necessary to trigger the filters. As long as the mean power in events is less than mean noise power, this will not make a significant difference.

[4] The estimate will be rudimentary because it does not reproduce background events arising from transient phenomena, which are intrinsically non-stationary and have higher-order correlations. It does, however, deal with the problem of estimating noise from a stationary non-Gaussian background.