Upper limits on burst sources of gravitational waves

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We present statistical methods for determining upper limit event rates on burst sources using gravitational wave detectors. By assuming that burst events are Poisson distributed in time, we devise two methods to estimate the event rate. The first is a generalization of the loudest event statistic used by Allen *et al*. [B. Allen *et al*., Phys. Rev. Lett. **83**, 1498 (1999)] to determine an upper limit on the number of binary neutron star inspirals in the Galaxy using data taken in 1994 using the LIGO 40-meter prototype interferometer at Caltech. If the probability distribution of background events can be determined, this method could be used to improve the event rate limit. However, in the absence of accurate knowledge of the distribution in signal to noise ratio of the false alarms, we show that the loudest event statistic determines a robust upper bound. The second method assumes that one sets a detection threshold before performing the data analysis, and that the number of events one observes with signal to noise exceeding this threshold is n. If the threshold has been chosen so that the probability of a false event is low, then this method determines a lower limit than the loudest event statistic. Nevertheless, we argue that the loudest event statistic provides a robust estimator, and, as such, is to be preferred when the distribution of signal to noise due to background events is not well understood.

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I. INTRODUCTION

Of the sources of gravitational waves that theorists have proposed, inspiralling binary neutron stars (or black holes) are likely to provide the strongest gravitational wave signals in the band of Earth based detectors. The next generations of kilometer-scale interferometers will be sensitive to binary neutron star inspirals out to a distance of tens to hundreds of Mpc. Given the uncertainties in event rate estimates for these sources, some of the first experiments using the new detectors will set upper limits on the rates directly from the gravitational wave data.

In a recent Letter, Allen *et al*. [1] described the analysis of approximately 45 hours of data taken with the LIGO 40-meter interferometer in November 1994. The detector was a prototype; correspondingly, the analysis was a prototype, intended principally to demonstrate an analysis pipeline aimed at determining an upper limit on the Galactic neutron star binary inspiral rate. This pipeline processed the raw interferometer data (along with calibration information) through a bank of matched filters designed to detect binary neutron star inspirals; the output of the pipeline was, for each segment of data and each filter in the bank, the peak signal to noise ratio (S/N) ρ recorded in the filter as well as a χ^2 statistic, which measured the time-frequency distribution of S/N of the peak filter output relative to that expected from a neutron star inspiral event. Large values of χ^2 indicate that the S/N is not accumulated in the manner predicted for binary inspiral signals. A threshold χ^2_* was chosen such that less than 10% of real signals would give $\chi^2 > \chi^2_*$ in the presence of stationary and Gaussian noise. Finally, the S/N was maximized over all of the filters in the bank that satisfied the χ^2 threshold to obtain a single maximum S/N for the segment of data.

To determine the upper limit, Allen *et al*. [1] focused on the single event with the greatest S/N over all the segments of data that passed all of the cuts in the analysis pipeline, i.e., the loudest event. Events with greater S/N are generally closer, and therefore rarer, than fainter events. Thus, in some fixed observation time, the probability that the loudest event has a particular amplitude is related to the event rate. The algorithm underlying the analysis is:

- Analyze the data through the data analysis pipeline,
- Set $\rho_{\max}$ equal to the amplitude of the loudest detected event or, if no events are detected, set $\rho_{\max} = 0$,
- Determine the efficiency with which the pipeline can detect binary neutron star inspiral events by Monte Carlo simulation.
- Compute an upper limit based on the number $\rho_{\max}$ and the efficiency of the pipeline at that S/N.

In this paper, we present statistical analyses that determine event rate (upper) limits for signals which are Poisson distributed in time, in the presence of an arbitrary noise background. In the limit that the loudest event is assumed to be a real signal, and the probability of a background event having S/N greater than or equal to that of the loudest event, this reproduces the limit presented in Ref. [1]. We use our formula to demonstrate the robust nature of the limit determined by Allen *et al*. This indicates that the new formula can be used to provide an improved upper limit if the background can be measured.

An alternative method to determine an upper limit event rate using data from the kilometer-scale interferometers might include the imposition of a threshold in S/N (as well as in χ^2). We derive the probability distribution for the event rate of real signals given the threshold and the number of events that are detected above the threshold. We compare this event rate limit, for several different outcomes of the experiment, with that obtained using the loudest event statistic.

In Sec. II, we introduce the notation and terminology that is used throughout the paper. We calculate, from first principles, the likelihood that the loudest event has magnitude $\rho_{\max}$ in Sec. III. This result has broad applicability in other contexts. Specializing to the assumptions made in [1], we arrive at the upper limit given there, and provide quantitative evidence of the robust nature of the limit. In Sec. IV, we derive an expression for the likelihood when using threshold detection in the data analysis pipeline. Finally, we present a critical assessment of the relative merits of each method.

II. NOTATION AND TERMINOLOGY

A data analysis pipeline process identifies *events*. Each identified event is characterized by a single number, its S/N ρ. We assume that discrete events are independent and are either *background* or *foreground* events. A background event is a *false alarm*: the signal being sought is not, in fact, present in the data stream at the time of the event. Background events may be caused by instrumental or environmental noise, or by "nuisance" signals (i.e., signals from other sources not being sought). Events corresponding to an actual signal are foreground events, and correspond to a *detection*. Not all signals give rise to foreground events: the analysis pipeline detects only some fraction of the signals. It is a bound on the rate of signals, which give rise to foreground events, that we wish to determine from the observations.

We assume that signal events are Poisson distributed with rate R, and that the analysis pipeline detects infinitesimally weak signal events in the absence of background events. Thus, the number of (Poisson distributed) foreground events in an observation of duration T is

$$P_f(N|\mu) = \frac{\mu^N e^{-\mu}}{N!} \tag{1a}$$

where

$$\mu = RT. \tag{1b}$$

Like all events, foreground events are each associated with a S/N ρ which characterizes their amplitude, and which depends on the analysis pipeline.* Denote the probability density that a foreground event is detected with a S/N ρ by $p_f(\rho)$:

$$p_f(\rho) = \left(\begin{array}{l} \text{probability that foreground} \\ \text{event has amplitude } \rho \end{array} \right) . \tag{1c}$$

Finally, denote the probability that a foreground event is detected with a S/N greater than ρ by $\epsilon(\rho)$ (the *efficiency* of detection for a given S/N):

$$\epsilon(\rho) = \int_\rho^\infty d\varrho \, p_f(\varrho) \tag{1d}$$

III. LOUDEST EVENT STATISTICS

In this section, we derive the posterior probability distribution for the event rate μ given a detection (either background or signal) with S/N $\rho_{\max}$, i.e. $P(\mu|\rho = \rho_{\max})$.

The probability that all foreground events, for a fixed rate μ, have S/N less than or equal to ρ is

$$P_f(\leq \rho|\mu) = P_f(0|\mu) + \sum_{N=1}^{\infty} P_f(N|\mu)[1-\epsilon(\rho)]^N \tag{2}$$

$$= \exp[-\mu\epsilon(\rho)] \, . \tag{3}$$

No assumptions are made about the distribution of signal to noise ratio for background events. We simply denote the (cumulative) probability that all background events, regardless of their number, have S/N less than or equal to ρ by $P_b(\rho)$. Finally, the probability of that an event, either foreground or background, has S/N less than or equal to ρ is

$$P(\leq \rho|\mu, b) = P_f(\leq \rho|\mu) P_b(\rho) \, . \tag{4}$$

We note an important point about the cumulative probability (3). Since the limit as $\rho \to 0$ is non-vanishing, i.e.,

$$\lim_{\rho\to 0} \Big[P_f(\leq \rho|\mu) \Big] = e^{-\mu} \, , \tag{5}$$

the associated probability density function is distributional at $\rho = 0$. In particular, the probability that an event has S/N ρ is

$$\begin{aligned} p(\rho|\mu, b) &= \frac{d}{d\rho} P(\leq \rho|\mu, b) \\ &= P_b(\rho) \Big[\mu p_f(\rho) e^{-\mu\epsilon(\rho)} + e^{-\mu}\Delta(\rho) \Big] \\ &\quad + e^{-\mu\epsilon(\rho)} p_b(\rho) \, , \end{aligned} \tag{6}$$

where $p_b(\rho) = dP_b(\rho)/d\rho$, and $\Delta(\rho)$ is a distribution† such that $\Delta(\rho) = 0$ when $\rho \neq 0$, and

$$\int_0^{\rho>0} d\varrho \, \Delta(\varrho) = 1 \, . \tag{7}$$

Using Bayes' law, we can write the probability (meaning degree of belief) the μ takes on a particular value as

$$P(\mu|\rho = \rho_{\max}) = \frac{p(\rho|\mu, b) p(\mu)}{p(\rho|b)} \tag{8}$$

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*The matched filtering S/N depends on the signal being filtered for, and on the power spectral estimate used in that filter.

†Since the integral in Eq. (7) is from zero to ρ, it is technically incorrect to call this a δ-function. Nevertheless, a physicist can understand it in those terms

where $p(\mu)$ is the *a priori* probability density for μ, and $p(\rho_{\max}|b)$ is the probability density for $\rho_{\max}$ integrated over all possible event rates:

$$p(\rho_{\max}|b) = \int_0^\infty d\mu\, p(\mu) p(\rho_{\max}|\mu, b) \ . \tag{9}$$

Assuming an improper uniform probability density $p(\mu)$, we can evaluate these expressions explicitly to get

$$P(\mu|\rho = \rho_{\max}) = \mu[\epsilon(\rho)]^2 e^{-\mu\epsilon(\rho)} \mathcal{N}(\rho) \tag{10}$$

where

$$\mathcal{N}(\rho) = \frac{P_b(\rho)p_f(\rho) + p_b(\rho)/\mu}{P_b(\rho)p_f(\rho) + p_b(\rho)\epsilon(\rho)} \ . \tag{11}$$

An upper limit on the event rate is determined by solving the equation

$$\begin{aligned} p &= \int_0^{\mu_p} d\mu\, P(\mu|\rho = \rho_{\max}) \\ &= 1 - e^{-\mu\epsilon} \left(\frac{\epsilon\mu_p P_b + P_b + \epsilon p_b/p_f}{P_b + \epsilon p_b/p_f} \right) \end{aligned} \tag{12}$$

for μ_p given the desired confidence p and the probabilities $\epsilon(\rho_{\max})$, $p_f(\rho_{\max})$, $P_b(\rho_{\max})$, and $p_b(\rho_{\max})$.

A. Weak upper limit calculation

In some circumstances, it is extremely difficult (if not impossible) to determine the false alarm probability $P_b(\rho)$ with any degree of confidence. Such a circumstance was encountered by Allen *et al*. [1], who made some assumptions to obtain a weak, but robust, upper limit on the event rate μ_p.

The assumptions in Ref. [1] were: (i) the loudest detected event from the data analysis pipeline is a signal, and (ii) the probability $P_b(\rho_{\max}) = 0$. Then, Eq. (12) reduces to

$$p = 1 - e^{-\mu_p\epsilon(\rho_{\max})}[1 + \mu_p\epsilon(\rho_{\max})] \ . \tag{13}$$

This implicit equation can be solved for $\mu_p\epsilon(\rho_{\max})$ given p; the results are shown in Fig. 1. In particular, the 90% confidence limit on the event rate is

$$\mu_{90\%} = \frac{3.890}{\epsilon(\rho_{\max})} \ . \tag{14}$$

As shown in Ref. [1], the function $\epsilon(\rho)$ can be determined by Monte Carlo injection of simulated signals (from the population to be constrained) into the data, and re-analyzing it. Using the data analysis pipeline described in Ref. [1], the S/N $\rho_{\max}$ of the loudest event was found to be $\rho_{\max} = 8.34$ and $\epsilon(\rho_{\max}) \simeq 0.33$ for neutron star binary inspiral signals from a Galactic population; this gives the rate limit $\mu_{90} = 11.788$ quoted in Ref. [1].

B. The weak upper limit is robust

The effects of the background are a major concern when the detector noise is not well understood. While one can assume, following Allen *et al*. [1], that the probability of a background event with S/N greater than or equal to $\rho_{\max}$ is zero, one would like to understand the errors that can arise from this assumption. Equation (12) can be used to quantify these errors by writing $\mu_p = 11.788$, and determining p as a function of $P_b(\rho_{\max})$ and $p_b(\rho_{\max})/p_f(\rho_{\max})$. If $p > 0.9$, the we say that the upper limit is robust. That is, with more information about the distribution $P_b(\rho)$, we could improve our limit on the event rate. Figure 2 demonstrates that the limit is robust when $\epsilon(\rho_{\max}) = 0.33$ and $\mu_{90\%} = 11.788$.

IV. THRESHOLD SEARCHES AND UPPER LIMITS

The hope for gravitational wave detection during the next decade is that we can understand the noise in the detectors by using the network of detectors at different locations around the world. A natural question, then, is how to set an upper limit when we have a good understanding of the noise in our detector. In this section, we explore this question in the context of a search for gravitational wave events with $\rho \geq \rho_*$ where $1 - P_b(\rho_*) \ll 1$. The output of the data analysis pipeline is the number of events n which exceed this threshold. Then the probability of interest is

$$p(\mu|n \text{ events with } \rho \geq \rho_*) = \frac{P(n \text{ events with } \rho \geq \rho_*|\mu)p(\mu)}{P(n \text{ events with } \rho \geq \rho_*)} \ . \tag{15}$$

First, the probability of detecting n events with $\rho \geq \rho_*$ given some event rate μ is

$$\begin{aligned} &P(n \text{ events with } \rho \geq \rho_*|\mu) \\ &\quad = \sum_{m=0}^{n} \binom{n}{m} P(n-m \text{ signals with } \rho \geq \rho_*|\mu) F_m \end{aligned} \tag{16}$$

where

$$F_m = P(m \text{ false alarms with } \rho \geq \rho_*) \ . \tag{17}$$

Note that $F_0 = P_b(\rho_*) \simeq 1$ and $\sum_{m=1}^{\infty} F_m = (1 - F_0) \ll 1$, so that $F_m \ll 1$ for all $m > 0$. Thus, we have the probability that n events with $\rho \geq \rho_*$ are recorded by the data analysis pipeline:

$$\begin{aligned} &P(n \text{ events with } \rho \geq \rho_*|\mu) \\ &\quad = \sum_{m=0}^{n} \binom{n}{m} F_m e^{-\mu\epsilon(\rho_*)} \frac{[\mu\epsilon(\rho_*)]^{n-m}}{(n-m)!} &(18) \\ &\quad \simeq F_0 e^{-\mu\epsilon(\rho_*)} \frac{[\mu\epsilon(\rho_*)]^n}{n!} \ . &(19) \end{aligned}$$

Clearly, this approximation should be valid whenever

$$F_m \ll F_0 \left[\frac{(n-m)!}{n!}\right]^2 [\mu\epsilon(\rho_*)]^m m!\ . \tag{20}$$

Using the improper uniform prior on μ as in the previous section, we arrive at our result

$$P(\mu < \mu_p | n \text{ events with } \rho \geq \rho_*) \simeq 1 - \frac{\Gamma[n+1, \mu_p\epsilon(\rho_*)]}{n!} \tag{21}$$

When $n = 0$ events are detected by the data analysis pipeline, the 90% confidence limit on μ is

$$\mu_{90\%} = \frac{2.303}{\epsilon(\rho_*)} \tag{22}$$

which would be 40% lower than that obtained using the loudest event statistic if $\rho_* = \rho_{\max}$.

As in the case of the loudest event statistic, it is important to ascertain the validity of the approximation that is made in Eq. (19). This can be done by examining Eq. (20) when $\mu = \mu_{90\%}$. Interestingly, the correction terms become more important as the number of detected events increases. In particular, we expect the approximation to be reasonable when $n \leq 2$.

V. CONCLUSIONS

We have derived a general and exact expression for the probability that the "loudest" event—i.e., the event of greatest signal-to-noise ratio—drawn with arbitrary efficiency from a population of Poisson distributed signals and in the presence of a Poisson distributed background takes on a particular value. If the background is assumed to be absent, the event rate based on the loudest event is 40% larger than one could hope to achieve by counting events louder than some pre-set threshold (that happens to be slightly larger than the loudest event) were the background truly absent; however, the loudest event statistic provides a *robust* rate limit when no knowledge of the background (and, therefore, no pratical way to choose a threshold) is available.

Improvements in the upper limit (still remaining in the framework of loudest event analyses) can be made by estimating the background event rate. For example, one could assume that the events between S/N of 6 and 7, shown in Fig. 2 of [1], are due to background and determine a power-law for the expected number of background events at higher S/N, allowing the assumption of no background events to be relaxed. Alternatively, Lazzarini [2] has suggested that a rudimentary estimate of the background rate could be obtained by evaluating the distribution of the detector output samples and, using that distribution together with an appropriate linear filter to reproduce the noise power spectrum, simulate and analyze detector output. Signals present in the original detector output will, by construction, not be present in the simulated output[‡]; consequently, only background events emerge at the end of the analysis pipeline when evaluated on the simulated data, allowing the background event rate to be estimated.[§]

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[1] B. Allen *et al*., Phys. Rev. Lett. **83**, 1498 (1999).
[2] A. Lazzarini, Private communication, 1999.

[‡] Their power will be, but without the correlations necessary to trigger the filters. As long as the mean power in events is less than mean noise power, this will not make a significant difference.

[§] The estimate will be rudimentary because it does not reproduce background events arising from transient phenomena, which are intrinsically non-stationary and have higher-order correlations. It does, however, deal with the problem of estimating noise from a stationary non-Gaussian background.

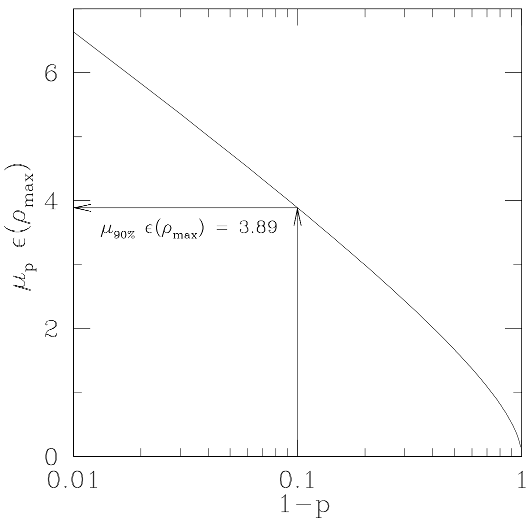

FIG. 1. The value of μ_p as a function of the confidence p computed with Eq. (13). The 90% confidence limit is indicated.

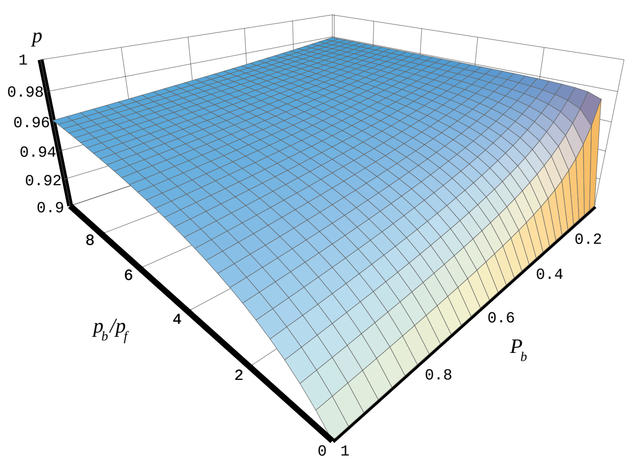

FIG. 2. The probability that $\mu \leq \mu_{90\%} = 11.788$ as a function of $P_b(\rho_{\rm max})$ and $p_b(\rho_{\rm max})/p_f(\rho_{\rm max})$. Since this probability is always greater than 0.9, we conclude that the weak limit discussed in Sec. III B is robust. For large p_b/p_f and $P_b \simeq 0$, p tends to $0.98 = 1-e^{-3.890}$, so the loudest event statistic (with a large, known background) gives the same limit as observing zero events above a fixed threshold in this limit.