Another derivation.....

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Sam claims that his analysis corresponds to the case where $N \geq 1$ and $\rho = \rho_{\rm max}$. So, we'll go through the derivation with this carefully in mind. We want to construct the posterior probability

$$P[\mu_0 | N \geq 1 \& \rho = \rho_{\rm max}] = \frac{p[N \geq 1 \& \rho = \rho_{\rm max} | \mu_0] p(\mu_0)}{p[N \geq 1 \& \rho = \rho_{\rm max}]} , \quad (1)$$

where

$$p[N \geq 1 \& \rho = \rho_{\rm max}] = \int \{ p[N \geq 1 \& \rho = \rho_{\rm max} | \mu_0] p(\mu_0) \} \, d\mu_0 , \quad (2)$$

and $p(\mu_0)$ is the prior on μ_0.

Use the general property of probabilities $P(A \& B) = P(A|B)P(B)$ to write

$$P[N \geq 1 \& \rho \leq \rho_{\rm max} | \mu_0] = P[\rho \leq \rho_{\rm max} | N \geq 1, \mu_0] P[N \geq 1 | \mu_0] . \quad (3)$$

The probability that the signal to noise ratio is less than or equal to $\rho_{\rm max}$ given at least one observed event is

$$\begin{aligned} P[\rho \leq \rho_{\rm max} | N \geq 1, \mu_0] &= \left\{ \sum_{N=1}^{\infty} P_0(N|\mu_0) c_0(\rho)^N \right\} / (1 - e^{-\mu_0}) \\ &= \left\{ \exp[-\mu_0(1 - c_0(\rho))] - \exp[-\mu_0] \right\} / (1 - e^{-\mu_0}) , \quad (4) \end{aligned}$$

and

$$P[N \geq 1 | \mu_0] = 1 - e^{-\mu_0} . \quad (5)$$

Now substitute Eqs. (4) and (5) into Eq. (3) to get

$$P[N \geq 1 \& \rho \leq \rho_{\rm max} | \mu_0] = \frac{\left\{ \exp[-\mu_0(1 - c_0(\rho))] - \exp[-\mu_0] \right\}}{(1 - e^{-\mu_0})} (1 - e^{-\mu_0}) . \quad (6)$$

Note that $P[N \geq 1 \& \rho \leq 0 | \mu_0] = 0$ as it should in this case. Finally, we can differentiate this with respect to $\rho_{\rm max}$ to get

$$p[N \geq 1 \& \rho = \rho_{\rm max} | \mu_0] = \mu_0 p_0(\rho_{\rm max}) \exp[-\mu_0(1 - c_0(\rho_{\rm max}))] \quad (7)$$

Using an improper uniform prior $p(\mu_0)$, we find that

$$P[\mu_0 | N \geq 1 \& \rho = \rho_{\rm max}] = \mu_0 [1 - c_0(\rho_{\rm max})]^2 e^{-\mu_0(1 - c_0(\rho_{\rm max}))} \quad (8)$$

which is identical to the right hand side of Eq. (19) of the notes by Brady and Creighton. The cumulative probability p that μ_0 is less than μ_p is then

$$p = 1 - e^{-\mu_p(1-c_0(\rho_{\rm max}))}[1 + \mu_p(1 - c_0(\rho_{\rm max}))] \; . \tag{9}$$

When $p = 0.9$, we arrive at the formula quoted by Allen et al., i.e.

$$R_{90\%} = \frac{3.890}{\epsilon(\rho_{\rm max})T} \; , \tag{10}$$

and the rate estimate $\mu_{90} = 11.788$.