Physics 717: Gravitation
Problem Set 1

1. Consider the manifold $\mathcal{M} = \mathbb{R}^3$. Using the chart based on Cartesian coordinates, construct the corresponding basis vectors X_μ of the tangent space at any point $p \in \mathcal{M}$.

2. Explicitly construct a cover of $\mathcal{M} = \mathbb{R}^3$ using spherical polar coordinates. How many charts are needed? Construct the basis vectors X_μ of the tangent space at any point $p \in \mathcal{M}$ based on the spherical polar chart. Explicitly construct the transformation between this basis and the basis arising in Problem 1.

3. Consider a piece of $\mathbb{R}^3$ defined by our classroom; the floor identifies a plane in $\mathbb{R}^3$. Consider the functions
$$f(p) = \text{height of } p \text{ above the floor}$$
and
$$g(p) = \text{distance squared from } p \text{ from one corner of the room .}$$
Construct $f \circ \psi^{-1}$ and $g \circ \psi^{-1}$ for the charts used in problems 1 and 2.

4. Consider a tensor T of type (k, l) over V. We construct a new tensor of type $(k-1, l-1)$ by *contraction* as follows:
$$C_{ij}T = \sum_{\sigma=1}^{n} T(\_, \ldots, v^{\sigma *}, \ldots \_; \_, \ldots, v_\sigma, \ldots, \_)$$
where $\{v_\sigma\}$ is a basis of V and $\{v^{\sigma *}\}$ is the dual basis. Show that $C_{ij}T$ is independent of the chosen basis $\{v_\sigma\}$ of V.

   [Hint: expand $\{v_\sigma\}$ over a new basis $\{v'_\sigma\}$, then use linearity and the definition of the dual basis to prove the result.]

5. Let X, Y, Z be smooth vector fields on a manifold M. Verify that their commutator satisfies the Jacobi identity:
$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \; .$$