Physics 717: Gravitation
Problem Set 3

1. Suppose the derivative operator ∇_a does not satisfy property (5) of our definition of derivative operators (i.e. the torsion free condition).

   (a) Let $\tilde{\nabla}_a$ be a torsion free derivative operator, then we know from class that

   $$\nabla_a w_b = \tilde{\nabla}_a w_b - C^c{}_{ab} w_c \ .$$

   Now let $\omega_b = \nabla_b f = \tilde{\nabla}_b f$ and prove that

   $$\nabla_a \nabla_b f = \nabla_b \nabla_a f + T^c{}_{ab} \nabla_c f$$

   where $T^c{}_{ab} = -T^c{}_{ba}$.

   (b) Starting from the definition of the commutator, i.e. $[X, Y](f) = X[Y(f)] - Y[X(f)]$, prove that

   $$T^c{}_{ab} X^a Y^b = X^a \nabla_a Y^c - Y^a \nabla_a X^c - [X, Y]^c \ .$$

2. For a flat 2-dimensional Euclidean space, the metric can be written as

   $$ds^2 = dx^2 + dy^2 = dr^2 + r^2\, d\theta^2$$

   where $\{x, y\}$ are the usual Euclidean coordinates and $\{r, \theta\}$ are the usual polar coordinates. Assuming that the geodesics are just straight lines, find the connection coefficients $\Gamma^\mu{}_{\alpha\beta}$ (in the polar coordinate system) using the geodesic equation

   $$\frac{d^2 x^\mu}{d\tau^2} + \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \Gamma^\mu{}_{\alpha\beta} = 0 \ .$$

3. For the 2-dimensional metric

   $$ds^2 = (-dt^2 + dx^2)/x^2$$

   (a) Find all connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$.

   (b) Write the timelike geodesic vector as $u^\alpha = dx^\alpha/d\tau$ where τ is the proper time along the geodesic. Show that the geodesic equations are obtained by varying the Lagrangian

   $$\mathcal{L} = \tfrac{1}{2} \left[-\left(\tfrac{dt}{d\tau}\right)^2 + \left(\tfrac{dx}{d\tau}\right)^2 \right] / x^2$$

   with respect to x and t.

   (c) Show that all timelike geodesics can be written in the form

   $$x^2 = (t/2 + C)^2 + A$$

   where A and C are real numbers.

   (d) Compute the Ricci tensor components $R_{\mu\nu}$ and show that

   $$R_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu} R$$

   where R is the Ricci scalar.