A Differential Form Short

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Abstract

Differential Forms provide a unique and alternative approach to computing the tensor fields of interest in general relativity. The technique is centered around totally anti-symmetric tensor fields. Using such fields, there is a clear and often straight forward procedure for deriving Ricci tensor, Riemann Curvature tensor and the like. This procedure gives a unique derivation for writing Einstein's equation in terms of a 2+2 and similar splits. The purpose of this paper is to introduce the definitions upon which the formalism is based and to use this formalism an example of a spacetime that is prevalent in current research in General Relativity.

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I. INTRODUCTION

It is standard in an introductory course on, and indeed often difficult to understand, General Relativity (GR) without introducing a coordinate basis. In all textbooks on GR, MTW[3], Weinberg[4], and Wald[2] for example, most of the derivations for Einstein's equations, the Connection, Riemann Curvature Tensor, etc. are written in terms of coordinate systems. We must recognize that which the coordinate basis method gives to us, the physicists. This method gives us the means to make calculations and thus predictions for GR. However, by no means does the coordinate basis method provide unique calculations in GR and in particular Einstein's equations. This method serves as one method of many, by which one can compute these equations, but the advantage of the coordinate basis method is the conceptual ease and thus the simplicity of introduction. This explains its prevalent use in introductory courses. What is less likely to appear in such a course but is almost always mentioned are different methods for deriving the important physical quantities of General Relativity, for example Ricci Tensor, Scalar Curvature, Covariant derivative, etc. . . .

One such method is the Differential Forms formalism. This formalism is a complete substitute for the traditional coordinate basis method commonly used in the literature. However, from a conceptual point of view, the coordinate basis method of computation and this formalism give identical results for Einstein's equations. As any GR student knows, the coordinate basis method for the calculation of Einstein's equations can sometimes be a difficult process. This is the point in which the differential forms formalism steps. The formalism can offer an alternative for the mindless derivations that come with choosing a coordinate basis and then using the Levi-Civita connection to get the Riemann curvature tensor. It introduces "frame" vectors, which are a set of linearly independent vectors that span the tangent space of the spacetime manifold at the particular point of interest, call it p. With these vectors, we write what are called the differential 1-forms corresponding to this set of vectors, by projecting these "frame" vectors onto a coordinate basis, upon which the spacetime metric's components are defined. Thus, it becomes possible to write the metric in terms of these differential forms. If we define certain maps, called the "wedge" product and exterior derivative, going from the vector space of these differential forms into itself, then it becomes possible to define and compute the curvature and thus formulate Einstein's equations. The unfortunate result of the differential forms formalism is that for a general metric, i.e. a metric that has no symmetry; this may offer no more computational ease than that of the coordinate basis method. However, if the metric has certain symmetries, for example those metrics which are spherically symmetric, then the differential forms formalism can save a significant amount of time in the calculation of the Einstein's equations associated with that symmetrical metric. The time that is saved comes from the introduction of the "frame" vectors that describe the orthogonal frame and using Cartan's Equations of structure. These serve to reduce the number of independent components of curvature and connections as compared to that of the coordinate basis method. Also, this method in no way aids one in finding solutions to Einstein's equations. The formalism is used purely for calculational purposes. A few examples, where this formalism is particularly useful, are the Vaidya and Robertson-Walker spacetime metrics.

In section two, we will go through a short general presentation of differential forms in GR. This serves to familiarize the reader with the terms, notations, and equations of this formalism. In section three, a 2+2 split of Einstein's equation for an arbitrary spherically symmetric spacetime is presented, where the differential forms formalism is used to calculate the Ricci and scalar curvature that are in Einstein's equations. In the last section, this split is applied to the specific example of a spherically symmetric retarded Vaidya spacetime.

I will adopt the following index notation throughout the paper

$$\begin{gathered} A, B, C, \ldots \in \{0,1\} \\ i, j, k, \ldots \in \{0,1\} \\ a, b, c, \ldots \in \{0,1,2,3\} \\ \alpha, \beta, \gamma, \ldots \in \{0,1,2,3\} \end{gathered} \tag{1}$$

where capital latin indices will represent frame 0 and 1 components, small latin indices in the first half of the alphabet will represent abstract vector index, greek indices will represent standard 4 dimensional basis components, and small latin indices from the latter half of the alphabet will represent coordinate 0 and 1 components.

II. DIFFERENTIAL FORMS

This section will not be a full treatment of differential forms. The reader should see W. Israel[1] for a more complete treatment of differential forms. A few results from Ref [1] will be stated here. There is no attempt by the writer to consider this a portion of the paper self-sufficient. So, we begin with the definition of differential forms. Differential forms are tensors that are anti-symmetrized over all of their indices. We call such tensors of rank (0,k) a k-form; e.g.

$$\phi(\tilde{dx}, \tilde{dy}) \equiv 2! \; F_{\alpha\beta} dx^{[\alpha} dy^{\beta]} \tag{2}$$

, is a differential form of rank 2. Thus a general k-form can be written

$$\phi\left(\tilde{dx^1}, \tilde{dx^2}, \ldots, \tilde{dx^k}\right) = k! F_{ab\ldots k} \; dx_1^{[a} dx_2^b \ldots dx_k^{k]} \tag{3}$$

. It is obvious from the definition above that a 1-form is a co-vector and that a 0-form is a scalar.

A. Wedge Product

There is a notion of how to form higher rank differential forms from lower rank forms. This notion is called the "wedge" product. The wedge product is a mapping $\wedge$: $V \oplus V \; \rightarrow V \otimes V$. Thus, the wedge product takes two differential forms, θ and ϕ, each rank k and n respectively and gives a differential (k+n)-form. This is written as follows

$$\theta \wedge \phi = (k+l)! T_{\alpha\beta\ldots\eta} P_{\sigma\tau\ldots\omega} \; dx_1^{[\alpha} dx_2^{\beta} \ldots dx_n^{\omega]} \tag{4}$$

where $T_{\alpha\beta\ldots\eta}$ and $P_{\sigma\tau\ldots\omega}$ are the tensor coordinate basis components.

B. The Exterior Derivative

The exterior derivative (d) is defined in the following way for a 1-form $\theta(\tilde{dx}) = 1! A_\alpha dx^\alpha$

$$d\theta(\tilde{dx}\ \tilde{dy}) = 2!\ \partial_\beta A_\alpha\ dx^{[\beta} \wedge dx^{\alpha]} \tag{5}$$

where we have used the fact that $d^2x = 0$. Notice that the exterior derivative of a 1-form is a 2-form, and in general differential forms of rank k are mapped into differential forms of rank k+1 by the exterior derivative. Also, that there is a straight forward generalization of taking exterior derivatives of higher rank differential forms. The exterior derivative also yields a generalized notion of the Leibnitz rule given in the following way

$$d\,(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\ \alpha \wedge d\beta \tag{6}$$

where k is the rank of α. This generalized Leibnitz rule will be used almost throughout without mention, but the reader is advised to keep this rule in mind. It is very important not to drop out any potential minus signs coming from the use of this rule.
In many situations that will be encountered later in this paper, the exterior derivative will be taken with respect to a 1-form, where in (5) the rank of α, a, will be zero. Thus, the $(-1)^a = 1$. It should also be noted that for an arbitrary k-form, Ω, that

$$d^2\Omega = 0 \tag{7}$$

. This will be used periodically throughout the paper.

C. Equations of Structure

The exterior derivative and the wedge product definitions are then put to use in the formulation of the equations of structure, which gives us equations for the connection 1-form and the curvature 2-form. We begin by letting M be our differential manifold. We choose a co-ordinate basis for a point $p \in M$. Pick a set of linearly independent vectors, $e_{(a)}(x^\mu)$, that span the tangent space of M at p. Notice that these linearly independent vectors are functions on the co-ordinate basis that we have chosen, and these linearly independent vectors serve to form a "frame" basis. So, we can expand this set of these vectors in terms of the co-ordinate basis and the components of this expansion will be called "the frame components" such that given any tensor $T_{ab\ldots}$, we can write the following

$$T_{ab\ldots} = T_{\alpha\beta\ldots} e^\alpha_{(a)} e^\beta_{(b)} \cdots \quad , \tag{8}$$

$$\theta^a = e^{(a)}_i\ dx^i. \tag{9}$$

where the θ^a will be called the θ-forms. Thus, given a spacetime metric $g_{\alpha\beta}$, there is a natural way of writing the frame components of this metric

$$g_{ab} = g_{\alpha\beta}\ e^\alpha_{(a)} e^\beta_{(b)}\ . \tag{10}$$

We use the frame components of the metric tensor to raise and lower frame indices as if they were our usual co-ordinate basis component indices. Thus these frame components of the metric tensor are the natural notion of the dual vector space of $e_{(a)}$, namely

$$e^{(a)} = g^{ab}\ e_{(b)} \tag{11}$$

. Thus, this immediately implies that

$$e^i_{(a)} e^{(a)}_j \; = \; \delta^i_j \tag{12}$$

. Now using Cartan's first equation of structure, which is gives us the relation between the connection 1-forms and the 1-forms specified by the frame vectors.

$$d\theta^a = -w^a_b \wedge \theta^b \tag{13}$$

where w_{ab} is the connection 1-form. Using the convention for g_{ab}; the following equations serve to fix the connection 1-form

$$\begin{aligned} g_{ab}\; d\theta^b \; &= \; C_{abc}\; \theta^c \wedge \theta^b \\ dg_{ab} \; &= \; D_{abc}\; \theta^c \end{aligned} \tag{14}$$

where $C_{abc} \; = -2 \; \gamma_{a[bc]}$, $D_{abc} \; = 2 \; \gamma_{(ab)c}$ and γ_{abc} are the ricci rotation coefficients. Thus the connection 1-form is

$$w_{ab} \; = \gamma_{abc}\; \theta^c \; = \; \frac{-1}{2} \left(-D_{abc} - D_{acb} + D_{bca} + C_{abc} + C_{bca} - C_{cab}\right) \theta^c. \tag{15}$$

This is one of the most important equations of the differential form formalism. This equation is where the symmetries of the spacetime are used to shorten the computation. Symmetric spacetimes have fewer non-zero independent connection 1-forms. All spacetimes allow a definition of a constant frame component metric and thus reducing the connection 1-form to be purely antisymmetric under interchange of its two indices, example Vaidya Spacetime (see section 4), which simplifies the equation (15) by setting $D_{abc} = 0 \forall a, b, c$.
Cartan's second equation of structure gives us the means to compute the Riemann curvature tensor in terms of the connection 1-forms and their exterior derivatives as follows

$$\Omega_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c_b = \frac{1}{2} R_{abcd}\; \theta^c \wedge \theta^d \; . \tag{16}$$

This equation will give the non-zero Riemann Curvature components. Now since Ω_{ab} is a differential form then it is antisymmetric under interchange of its two indices. Likewise from Cartan's Second Equation of Structure that exchanging the c and d indices on the wedged 1-forms as introduces a minus sign. Thus, by calculating the 6 curvature 2-form frame components from (15), we have actually computed all of the non-zero Riemann Curvature components, where

$$\begin{aligned} R_{abcd} &= -R_{bacd} \\ R_{abcd} &= -R_{abdc} \end{aligned} \; . \tag{17}$$

Once, you have the Riemann curvature tensor components, then you are home free. Just calculate the Einstein tensor and set it proportional to the stress-energy tensor.

III. 2+2 SPLIT OF GENERAL RELATIVITY

In this section, I present a 2+2 split of Einstein's equations for a generalized spherically spacetime metric using a differential forms formalism.
Consider the following invariant line element

$$ds^2 = \gamma_{ij} dx^i dx^j + r^2(x^i) d\Omega^2 \tag{18}$$

where $d\Omega$ is the usual solid angle, γ_{ij} is a symmetric 2x2 matrix and is a function of x^i only. Recall from (1), that the indices $i, j \in (0,1)$.
Now we proceed with the usual approach of generating Einstein's equations using differential forms. First, we choose frame components for our differential forms that correspond 0 and 1 components of our metric and distinguish them with A,B frame indices (1).
So, let

$$\begin{array}{ll} \theta^A = e_i^{(A)}\ dx^i & dx^i = e^i_{(A)}\ \theta^A \\ \theta^2 = r\left(x^i\right)\ d\theta & d\theta = \frac{1}{r(x^i)}\ \theta^2 \\ \theta^3 = r\sin\theta\ d\phi & d\phi = \frac{1}{r\sin\theta}\ \theta^3 \end{array} \tag{19}$$

be the differential forms and inverses for the given metric. The θ^a are the θ forms from section two. Also, we define the frame vectors such that it is possible to write g_{AB} with the following frame components

$$g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{20}$$

With these definitions, it is easily found from the definition of the exterior derivative that

$$d\theta^A = \gamma^A_{BC}\ \theta^B \wedge \theta^C, \tag{21}$$

$$d\theta^2 = \frac{D_A r}{r}\ \theta^A \wedge \theta^2, \tag{22}$$

$$d\theta^3 = \frac{D_A r}{r}\ \theta^A \wedge \theta^3 + \frac{(\cot\theta)_2}{r}\theta^2 \wedge \theta^3. \tag{23}$$

where $D_A r$ is defined to be

$$D_A r = r_{,i}\ e^i_A \tag{24}$$

and will be called the frame derivative.
Now, we will use Cartan's equations of structure to find the connection 1-forms, ω_{ab} and the curvature 2-forms, Ω_{ab}. These results will be used to derive Einstein's equations, which should give us the explicit formalism for the 2+2 split. So g_{ab} constant and equation (14) implies that $\gamma_{(ab)c} = 0\ \forall\ a, b, c$; and thus equation (15)is sufficient to determine ω_{ab} uniquely. They are found to be

$$\omega_{AB} = \gamma_{ABC}\theta^C, \tag{25}$$

$$\omega_{A2} = -\frac{D_A r}{r}\ \theta^2, \tag{26}$$

$$\omega_{A3} = -\frac{D_A r}{r}\ \theta^3, \tag{27}$$

$$\omega_{23} = -\frac{(\cot\theta)_2}{r}\ \theta^3. \tag{28}$$

Also, since the ω_{ab} is purely antisymmetric, we thus know that $\omega_{ab} = -\omega_{ba}$. As a consequence, we know ω_{BA} , ω_{2A} , etc.... From the definition of the exterior derivative, we know the exterior derivative of the connection one forms.

$$d\omega_{A2} = -\frac{D_B\left(D_A r\right)}{r}\ \theta^B \wedge \theta^2, \tag{29}$$

$$d\omega_{A3} = -\frac{D_B\left(D_A r\right)}{r}\ \theta^B \wedge \theta^3 - \frac{D_A r\ \cot\theta}{r^2}\ \theta^2 \wedge \theta^3, \tag{30}$$

$$d\omega_{23} = \frac{1}{r^2}\ \theta^2 \wedge \theta^3. \tag{31}$$

Now, we make use of Cartan's second equation of structure. It is this equation that gives us a way of calculating the Riemann curvature tensor and thus the Ricci and Scalar curvature components. Of course, these tensors are of highly important physical significance because these will be plugged directly into Einstein's equation. So, using Cartan's second equation of structure (16), the non-zero Curvature two form components can be written as follows

$$\Omega_{AB} = d\omega_{AB} + \omega_{AC} \wedge \omega^C_B, \tag{32}$$

$$\Omega_{A2} = -\frac{1}{r}\left(D_C\left(D_A r\right) + D^B r\ \gamma_{ABC}\right)\ \theta^C \wedge \theta^2, \tag{33}$$

$$\Omega_{A3} = -\frac{1}{r}\left(D_C\left(D_A r\right) + \gamma_{ABC} D^B r\right)\ \theta^C \wedge \theta^3, \tag{34}$$

$$\Omega_{23} = \frac{1}{r^2}\left(1 - D_B r\ D^B r\right)\ \theta^2 \wedge \theta^3. \tag{35}$$

Since, Ω_{ab} is a 2-form, then we know that it is anti-symmetric and thus it is completely known.
Reading off the independent non-zero Riemann curvature components from equations (32-35) using (16) gives

$$\begin{aligned} R_{ABCD} &= {}^{(2)}R_{ABCD}, \\ R_{A2C2} &= -\tfrac{2}{r}\left(D_C\left(D_A r\right) + D^B r\ \gamma_{ABC}\right) \\ R_{A3C3} &= R_{A2C2}, \\ R_{2323} &= \tfrac{2}{r^2}\left(1 - D_B r\ D^B r\right) \end{aligned} \tag{36}$$

From equation (16), we can then immediately read off all of the non-zero Riemann Curvature Components. Thus, the Riemann Curvature Tensor is known to us, and now the non-zero Ricci Tensor components can be easily calculated and are found to be

$$R_{AB} = {}^{(2)}R_{AB} - 4\left(\frac{D_B\left(D_A r\right)}{r} + \frac{D^C r}{r}\ \gamma_{ACB}\right), \tag{37}$$

$$R_{22} = -\ \frac{2}{r}\left(D^A\left(D_A r\right) + D^B r\ \gamma_{AB}{}^A\right), \tag{38}$$

$$R_{33} = R_{22}, \tag{39}$$

Contracting the Ricci Tensor to get the scalar curvature

$$R =^{(2)} R - \ \frac{4}{r}\left(D^A\left(D_A r\right) + D^B r\ \gamma_{AB}{}^A\right) - 4\left(\frac{D_A\left(D_A r\right)}{r} + \frac{D^C r}{r}\ \gamma_{AC}{}^A\right), \tag{40}$$

this then gives the non-zero Einstein Tensor components as follows

$$G_{AB} = {}^{(2)}G_{AB} + \ \frac{2}{r}\left(D_B\left(D_A r\right) + D^C r\ \gamma_{ACB}\right), \tag{41}$$

$$G_{22} = -\frac{1}{2}\ g_{AB}\ {}^{(2)}R + 2\left(\frac{D_A\left(D_A r\right)}{r} + \frac{D^C r}{r}\ \gamma_{AC}{}^A\right), \tag{42}$$

$$G_{33} = G_{22}, \tag{43}$$

where ${}^{(2)}G_{AB}$ is the Einstein tensor of the 2 dimensional hypersurface corresponding to the zero and one frame components. This is the 2+2 split of Einstein's equation. We have broken the calculation of Einstein's equations into two calculations. The first of these calculations accounts for the curvature on the AB frame component slice and the other calculation calculates the contribution of the curvature on the angular slice.

IV. VAIDYA SPACETIME

Lets apply this technique to a simple example. Consider the following Vaidya Spacetime metric

$$ds^2 = -f du^2 + 2dudr + r^2 d\Omega^2 \tag{44}$$

where $f = 1 - \frac{2m(u)}{r}$ and $d\Omega$ is the usual solid angle. Let

$$\begin{aligned} du &= dx^0 \\ dr &= dx^1 \\ d\theta &= dx^2 \\ d\phi &= dx^3. \end{aligned} \tag{45}$$

Also, lets choose θ^a, the differential 1-forms for the Vaidya metric, as follows

$$\begin{array}{cc} \theta^0 = du & \theta^1 = \left(dr - \frac{f}{2}du\right) \\ \theta^2 = rd\theta & \theta^3 = r\sin(\theta)\, d\phi \end{array} \tag{46}$$

So, equation (44), the invariant line element ds^2, becomes

$$ds^2 = 2\theta^0\theta^1 + \left(\theta^2\right)^2 + \left(\theta^3\right)^2 \tag{47}$$

Notice that defining the 1-forms in this way gives simple non-zero orthogonal vectors which constitute the frame $e_i^{(a)}$

$$\begin{array}{ccc} e_1^{(1)} = 1 & e_0^{(1)} = -\frac{f}{2} & e_0^{(0)} = 1 \\ e_2^{(2)} = r & e_3^{(3)} = r \ \sin\theta & \end{array}, \tag{48}$$

and the corresponding non-zero dual vectors are

$$\begin{array}{ccc} e^1_{(0)} = 1 & e^1_{(0)} = frac f2 & e^1_{(1)} = 1 \\ e^2_{(2)} = r^{-1} & e^3_{(3)} = (r\sin(\theta)) & \end{array}. \tag{49}$$

Thus, the frame components of the metric are written

$$g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

Also, note that this corresponds to a specific example of metric that was specified in section 3. Only here

$$\begin{array}{c} \gamma_{ij} = \begin{pmatrix} -f & 1 \\ 1 & 0 \end{pmatrix} \\ D_A r = \begin{cases} 1 & \leftrightarrow A = 1 \\ \frac{f}{2} & \leftrightarrow A = 0 \end{cases} \end{array} . \tag{51}$$

Taking the exterior derivative of this is obviously zero. So, the connection 1-form will be purely anti-symmetric, i.e. $\omega_{ab} = -\omega_{ba}$. Also, taking the exterior derivative of the 1-forms θ^a or using equations (21)-(23)

$$\begin{array}{rcl} d\theta^0 &=& 0 \; , \\ d\theta^2 &=& \frac{1}{r}\theta^1 \wedge \theta^2 + \frac{f}{2r}\theta^0 \wedge \theta^2 \; , \\ d\theta^1 &=& -\frac{m(u)}{r^2}\theta^1 \wedge \theta^0 \; , \\ d\theta^3 &=& \frac{1}{r}\theta^1 \wedge \theta^3 + \frac{f}{2r}\theta^0 \wedge \theta^3 + \frac{\cot(\theta)}{r}\theta^2 \wedge \theta^3 \; . \end{array} \tag{52}$$

Now from equation (14), the Ricci rotation coefficients are known immediately. From equation (15) or equations (21)-(23), it can be inferred that the non-zero independent connection 1-form frame components are

$$\begin{array}{ccc} \omega_{01} = -\frac{m(u)}{r^2}\theta^0 & \omega_{02} = -\frac{f}{2r}\theta^2 \\ \omega_{03} = -\frac{f}{2r}\theta^3 & \omega_{12} = -\frac{1}{r}\theta^2 \\ \omega_{13} = -\frac{1}{r}\theta^3 & \omega_{23} = -\frac{cot(\theta)}{r}\theta^3 \end{array} . \tag{53}$$

Using equations (32)-(35), we have a general derivation for the Riemann Curvature Tensor components, where $D_A r = 1$ if A=1 or $\frac{f}{2}$ if A=0. Thus, we get the following expression for the curvature 2-form frame components

$$\begin{array}{c} \Omega_{01} = \frac{2m(u)}{r^3} \\ \Omega_{02} = -\frac{m(u)}{r^3}\theta^1 \wedge \theta^2 + \frac{m'(u)}{r^2}\theta^0 \wedge \theta^2 \\ \Omega_{03} = \left(\frac{m'(u)}{r^2} + \frac{fm(u)}{r^3}\right)\theta^0 \wedge \theta^3 + \frac{m(u)}{r^3}\theta^1 \wedge \theta^3 + \frac{f\cot(\theta)}{r^2}\theta^2 \wedge \theta^3 \\ \Omega_{12} = \frac{m(u)}{r^3}\theta^2 \wedge \theta^0 \\ \Omega_{13} = \frac{m(u)}{r^3}\theta^3 \wedge \theta^0 \\ \Omega_{23} = \frac{2m(u)}{r^3}\theta^2 \wedge \theta^3 \end{array} . \tag{54}$$

Using the Cartan's second equation of structure, eq(15); we get the following non-zero independent Riemann Curvature components

$$\begin{array}{ccc} R_{0110} = 4\frac{m(u)}{r^3} & R_{0212} = -2\frac{m(u)}{r^3} & R_{0202} = 2\frac{m'(u)}{r^2} \\ R_{0313} = 2\frac{m(u)}{r^3} & R_{0323} = 2\frac{f\cot(\theta)}{r^2} & R_{1220} = 2\frac{m(u)}{r^3} \\ R_{1330} = 2\frac{m(u)}{r^3} & R_{2323} = 4\frac{m(u)}{r^3} \end{array} . \tag{55}$$

All of the other components of the Riemann Curvature tensor can be obtained by noting, as mentioned in section 3. Thus, there are a total of 27 non-zero components of the Riemann Curvature tensor.

V. CONCLUDING REMARKS

The differential forms formalism does make certain metrics easier to handle in terms of deriving the curvature, but there is one point that I found can be most troublesome. When one is using this method to calculate curvature, minus signs are often very easily dropped from the derivations. On the other hand though, checking one's derivations usually only takes a few minutes. Thus, although it is very easy to drop a minus sign; it is just as easy to check the result by another derivation to find the dropped minus sign. Other than that, the derivations themselves involve much less algebra than the coordinate basis method and this makes for simpler to check derivations. The formalism also allows for simple ways of writing the 2+2 split. It is written in terms of frame derivatives, $D_a(f)$; unlike the co-ordinate basis method whose formulation is not so simple looking and in general much more difficult to calculate. It is possible to teach oneself this method in a short time of about an hour. It will take sometime and patience to learn to get the minus signs correct, but other than that the method will be known.

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