Gravitational collapse and spacetime singularities

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The nature of gravitational collapse was the subject of debate and controversy for much of the last century. During the 1960's and 1970's our understanding of general relativity consolidated and culminated in our modern understanding of *black holes* and spacetime singularities. Since then, there has been significant advances in understanding the dynamics of gravitational collapse and the nature of singularities that arise from collapse. This series of 5 lectures were presented at the X Brazillian School on Cosmology and Gravitation. The first lecture presents a detailed discussion of Schwarzschild spacetime and Oppenheimer-Snyder collapse. The subsequent lectures cover perfect-fluid collapse, critical phenomena, the relevance of self-similarity, cosmic censorship and finally the nature of singularities inside charged/rotating black holes. The presentation is intended as a starting point for students wishing to pursue research in general area of gravitational collapse and singularities. It is not mathematically rigorous.

I. LECTURE 1: PRELIMINARIES

When Einstein (1) published his vacuum field equations in 1915 he had already derived from them the advance of the perihelion of Mercury using a difficult perturbative approach. The following year Karl Schwarzschild (2) discovered the exact solution to the field equations representing the external gravitational field of a static, spherically symmetric body. This solution allows a very simple derivation of the perihelion advance and was the first black hole solution to Einstein's equations discovered. At the time, however, neither the term nor the concept of a black hole was known.

During the following forty years advances in the relativistic theory of massive stars were rather slow. In 1931 it was noted, by Chandrasekhar (4) and independently by Landau (5), that there was an upper limit to the mass which a cold star could have and continue to support itself against Newtonian gravity. Both Eddington and Landau realized that any star above the Chandrasekhar limit would, upon using all of its nuclear fuel, contract dramatically producing an object from which light could not even escape (5; 6). However, there was great reluctance to accept that this could occur in nature. Einstein himself was vehemently opposed to the idea of stellar collapse, and in 1939 he constructed model solutions to his field equations which showed that *stationary* configurations of matter with mass M must have a radius $R > 2GM/c^2$ (7). That same year however, Oppenheimer and Volkoff (8) completed calculations showing that there was an upper limit to the mass of a star in General Relativity too. In subsequent work with Hartland Snyder (9), Oppenheimer went on to examine the gravitational collapse of a massive star to zero volume. This work now serves as the paradigm of gravitational collapse.

It was during the 1960's and early 1970's that most of our present knowledge of black holes was obtained. Advances in the understanding of classical processes involved in gravitational collapse were dramatic. A series of theorems (see chapter 12 of (10) for a review) established the uniqueness of stationary black holes, showing that their only external attributes are mass, angular momentum and electric charge; Wheeler used the graphic phrase "black holes have no hair" to summarize these results. New global methods were employed to prove the singularity theorems (11; 12) and to derive other properties of black holes (12).

In this lecture, I present the field equations for spherically symmetric spacetimes with an arbitrary matter source. I then review the Schwarzschild solution and the series of coordinate transformations which lead to the analytic extension of spacetime through the event horizon. Finally, I discuss the nature of the event horizon and the singularity in the Schwarzschild solution.

A. Einstein equations in spherical symmetry

In this section, we derive the Einstein equations in the presence of an arbitrary stress-energy tensor $T_{\mu\nu}$. (The matter equations of motion will be discussed as needed below.) The starting point is the line element

$$ds^2 = -\Phi^2 f dt^2 + f^{-1} dr^2 + r^2\, d\Omega^2 \quad (1)$$

where $\Phi = \Phi(r,t)$ and $f = f(r,t) = 1 - 2m(r,t)/r$. Using the results from appendix **??**, we derive three independent equations governing the metric functions:

$$\frac{\partial \Phi}{\partial r} = 4\pi r f^{-1} \left(T^r{}_r - T^t{}_t \right) , \quad (2)$$

$$\frac{\partial m}{\partial r} = -4\pi r^2 T^t{}_t , \quad (3)$$

$$\frac{\partial m}{\partial t} = 4\pi r^2 T^r{}_t . \quad (4)$$

It is worthwhile to comment on the physical interpretation of these equations. According to Eq. (3), the function $m(r,t)$ measures the amount of energy inside radius r at constant time t. Moreover, Eq. (4 says that the flux of stress-energy across a

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*Electronic address: patrick@gravity.phys.uwm.edu sphere of radius r determines the change in $m(r,t)$ as a function of time t. For this reason, we refer to $m(r,t)$ as the mass interior to a sphere of radius r at time t. In asymptotically flat spacetimes, this function agrees with both the ADM and Bondi masses at spacelike and null infinity, respectively, thus lending support to the interpretation as mass.

B. Schwarzschild solution

Consider the situation in which spacetime is vacuum, that is $T_{\mu\nu} = 0$ everywhere. Equations (2)-(4) can now be easily solved to find $m =$ constant and $\Phi = \Phi(t)$. Since Φ appears in the line element multiplying dt, it is clear that a simple modification of the time coordinate can be used to make $\Phi = 1$. Thus, we arrive at the standard form of the Schwarschild solution

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

$$f = 1 - \frac{2m}{r} . \tag{6}$$

In the limit $r \to \infty$, notice that this solution approaches the Minkowski line-element; it is asymptotically flat.[1] We also notice that the metric is singular on the surface $r = 2m$. This singularity was the source of much confusion in the early history of general relativity; it is studied in detail below.

1. Eddington-Finkelstein coordinates

The light-cone structure of spacetime is critical to understanding the physical nature of any solution of Einstein's equations. Eddington-Finkelstein coordinates facilitate the study of null rays in Schwarzschild spacetime. Introduce advanced Eddington-Finkelstein time by

$$dv = dt + \frac{dr}{f} \tag{7}$$

and rewrite the line element as

$$ds^2 = -f dv^2 + 2dv\, dr + r^2 d\Omega^2 . \tag{8}$$

We note that the surface $r = 2m$ is no longer problematic– the first term in the line-element vanishes, but the remaining terms are finite and the metric is non-degenerate. This is our first hint that the singularity at $r = 2m$ is not physical.

In these coordinates, the radial lightlike geodesics satisfy

$$-f\left(\frac{dv}{d\lambda}\right)^2 + \frac{dv}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{9}$$

where λ is an affine parameter along the geodesic. The ingoing geodesics are given by $v =$ constant, while the outgoing null geodesics satisfy the differential equation:

$$\frac{dr}{dv} = \frac{f}{2} . \tag{10}$$

Notice that $r = 2m$ is a trivial solution of Eq. (10). This shows that the surface $r = 2m$ is generated by outgoing radial null-rays – it is a null hypersurface. In (r,v)-space, these radial null geodesics separate spacetime into two distinct regions:

1. $\mathbf{r > 2m}$ – the radius r increases along outgoing null rays since $f > 0$.
2. $\mathbf{r < 2m}$ – the radius r decreases along outgoing null rays since $f < 0$.

It is straightforward to integrate Eq. (10) to find the implicit relation

$$v = 2(r - r_0) + 4m \ln\left|\frac{r - 2m}{r_0 - 2m}\right| \tag{11}$$

where an integration constant has been chosen so that $r = r_0$ when $v = 0$. Examples of radial null geodesics are plotted in Fig. **??**. Notice that $v \to \infty$ as $r \to \infty$ along outgoing lightrays which have $r_0 > 2m$. If, instead, $r_0 < 2m$ the radius is a decreasing function of v and $r \to 0$ at some finite advanced time. For an observer at large radius, only the geodesics that originate from $r_0 > 2m$ can be received and the region $r \leq 2m$ is inaccessible to direct observation. The hypersurface $r = 2m$ is an *event horizon*, and the region $r < 2m$ is a black hole.[2] Finally, notice that $r \to 2m$ as $v \to -\infty$ all outgoing null rays; this will be important later.

2. Maximal analytic extension of Schwarzschild spacetime

The system of coordinates (v,r) is singular on the past event horizon: both r and v are constant there (v is actually negative infinity). It is possible to construct coordinates which are regular on both the future and past horizons of the black hole. Introduce the coordinate

$$u = v - 2\int \frac{dr}{f} = v - 2r - 4m \ln\left|\frac{r - 2m}{2m}\right| \tag{12}$$

which is infinite when $r \to 2m$ along $v =$ constant. This is Eddington-Finkelstein retarded time. Moreover the coordinates (u,v) cover the region $2m < r < \infty$. In terms of new coordinates (U,V) defined by

$$V = e^{\kappa_e v} , \tag{13}$$

$$U = -e^{-\kappa_e u} , \tag{14}$$

[1] In general, it is not enough to simply have the metric approach Minkowski form. One should check that the curvature vanishes fast enough in this limit.

[2] The rigorous definition of a black hole in asymptotically flat spacetime makes reference to future null infinity, not an observer that remains at large radius. The general idea is the same, however.

where $\kappa_e = 1/4m$, the line element (**??**) takes the form

$$ds^2 = \frac{f}{\kappa_e^2 UV}\, dU dV + r^2 d\Omega^2 \, . \tag{15}$$

To see that the metric is indeed regular on the inner horizon combine Eqs. (12)–(14) to get

$$UV = -\frac{r-2m}{2m} e^{r/2m} \, , \tag{16}$$

from which it is easy to see that the line-element becomes

$$ds^2 = -\frac{32m^2}{r} e^{-r/2m}\, dU dV + r^2 d\Omega^2 \, . \tag{17}$$

Notice that $-\infty < U < 0$ and $0 < V < \infty$ when $r > 2m$. To analytically continue through the horizon, simply allow U and V to range over all real values. These coordinates then cover the entire Schwarschild spacetime $0 \leq r < \infty$ – they are Kruskal coordinates. The conformal diagram for Schwarschild spacetime in Fig. **??** shows values of the various coordinates on the event horizons and on future null infinity.

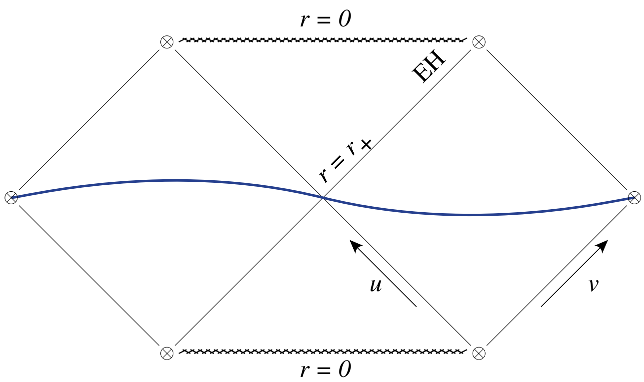

FIG. 1 Stuff about black holes

3. Timelike geodesics

An analysis of timelike geodesics in this spacetime gives us some idea of the intrigueing global structure of this spacetime. The equation of a radial timelike geodesic can be reduced to

$$\dot{r}^2 = E^2 - f \, . \tag{18}$$

Let us suppose the energy satisfies $|E| > 1$, although a similar analysis can be done for the other cases. The right hand side of (35) can become zero at a finite radial value

$$r_b = \frac{-m + \sqrt{m^2 + (E^2-1)e^2}}{(E^2-1)} < r_i \, . \tag{19}$$

Therefore an observer who falls into the black hole decelerates until he reaches the finite radius r_b at which he comes to rest. Subsequently the same observer moves in the direction of increasing r and can return to arbitrarily large radii in a finite proper time. Thus it seems that it is possible to fall into a charged black hole, to avoid hitting the curvature singularity at $r = 0$ and to return to large radii again (see Fig. IV.A.2). Clearly this is very different to the behavior of geodesics in Scwharzschild spacetime (see section **??** in the Introduction).

C. Singularity at $r = 0$

D. Lightlike congruences

Write some stuff about the congruence of lightlike geodesics orthogonal to the 2-spheres.

II. LECTURES 2 & 3: DYNAMICS OF GRAVITATIONAL COLLAPSE

In these lectures, I discuss features of gravitational collapse in general relativity. For concreteness, attention is focused on the collapse of perfect fluids in spherical symmetry. The starting point is Oppenheimer-Snyder collapse of a uniform density ball of dust which has serves as the paradigm for gravitational collapse in general. In general, the initial density and initial velocity profile may be chosen in perfect fluid collapse. The richness of solutions to general relativistic gravitational collapse was clearly demonstrated in Choptuik's original study of critical phenomena in scalar field collapse. The evolution of one-parameter families of initial data are used to demonstrate critical phenomena in gravitational collapse of perfect fluids. The relevance of self-similarity in near critical evolutions and the origin of black hole mass scaling are explained.

In this lecture, I consider the gravitational collapse of perfect fluid as an example of black hole formation.

- Stress-energy tensor for fluid
- Covariant conservation
- Dust collapse – Oppenheimer-Snyder
- Initial data for collapse
- Critical phenomena and mass scaling
- Critical phenomena and self-similarity
- Explanation of the critical exponent

A. Spherical perfect fluids

The stress-energy of a perfect fluid with 4-velocity u^μ is

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \, . \tag{20}$$

Here, ρ is the energy density of the fluid as measured by an observer moving with a fluid element, and p is the tangential pressure. The equations of motion for the fluid are determined by covariant conservation of the stress-energy $T_\mu{}^\nu{}_{;\nu} = 0$, when supplemented with an equation of state for the fluid, which we take to be $p = \gamma\rho$. The constant of proportionality γ is restricted to the range $0 \leq \gamma \leq 1$. The upper limit is set by causality; the speed of sound in the fluid must be less than or equal to that of light. Finally, the four velocity u^μ is a timelike vector, *i.e.* $u^\mu u_\mu = -1$, so that its components can be expressed in terms of the three velocity V as

$$u^t = \Phi f^{1/2}/\sqrt{(1-V^2)}\ , \quad u^r = f^{-1/2}V/\sqrt{(1-V^2)}\ . \tag{21}$$

It turns out that these equations are conveniently cast as a system of conservation laws in terms of the variables $x = T_t{}^t$ and $y = \Phi f T_r{}^t$. These quantities have a direct physical interpretation as the energy-density measured by an observer at fixed radius and the momentum flux across an $r =$ constant surface, respectively. The evolution equations can now be written as

$$\frac{\partial\,\mathbf{x}}{\partial t} + \frac{\partial\,\mathbf{F}(\mathbf{x}, r)}{\partial r} = \mathbf{S} \tag{22}$$

where $\mathbf{x} = (x, y)^T$, and the flux $\mathbf{F}$ is explicitly

$$\mathbf{F} = \begin{bmatrix} -\Phi f y \\ \Phi f\{(\gamma-1)\rho - x\} \end{bmatrix}\ . \tag{23}$$

The two components of the vector $\mathbf{S} = (S_1, S_2)$ are

$$S_1 = 2\Phi f y/r \tag{24}$$

$$S_2 = -8\pi r\gamma\rho^2\Phi + \frac{(\gamma-1)\rho\Phi(1-f)}{2r} + \frac{2(\rho+x)f\Phi}{r} \tag{25}$$

In these equations, ρ is a functional of x and y determined explicitly by

$$\rho = \frac{-x(\gamma-1) \pm \sqrt{x^2(\gamma-1)^2 + 4\gamma(x^2-y^2)}}{2\gamma}\ . \tag{26}$$

By considering the limit of zero three velocity, $y \to 0$, the positive sign is determined to be the appropriate one. The square root in this expression caused significant problems during the numerical evolutions in which γ was close to unity; the *ad hoc* solution adopted by us is discussed in section XX.

Two further independent equations are provided by the Einstein equations, both of which involve only radial derivatives. The mass-function satisfies

$$m' = -4\pi r^2 x\ , \tag{27}$$

while Φ is determined by

$$\Phi' = 4\pi r f^{-1}\Phi[(\gamma-1)\rho - 2x]\ . \tag{28}$$

These are first order, ordinary differential equations on each time slice. Regularity of the origin requires that the mass satisfy $m(0, t) = 0$, at least up until a singularity is encountered. We also normalise t to be proper time for an observer at $r = 0$, thus we set $\Phi(0, t) = 1$.

Finally, we note that spherical symmetry also implies that the three velocity of the fluid must vaniish at $r = 0$. This leaves only the central energy density to be determined during the evolution.

B. Collapse of spherical ball of dust

To understand the process of collapse it is only necessary to study the motion of the stellar surface located at $r = R(\tau)$ where τ is the proper time along the timelike geodesic it follows. The equation of motion of this surface is simply

$$\dot{R}^2 = -(1 - \frac{2m}{R}) + E^2\ , \tag{29}$$

where E is a constant and a dot represents differentiation with respect to proper time. Integrating this equation shows that a star can collapse to $R = 2m$, the gravitational radius of the system, in a finite proper time. In fact, once it reaches this radius the surface of the star must continue to $R = 0$. The time of free-fall, as measured by an observer on the surface of the star, from the gravitational radius to the origin is approximately one day for a star of $\sim 10^{10} M_\odot$.

Thus, the star collapses to zero radius in a finite proper time. For an external observer things appear different, however. A long way from the black hole t is the proper time along an observer's path. The motion of the stellar surface parameterized by this time

$$\left(\frac{dR}{dt}\right)^2 = \left(1 - \frac{2m}{R}\right)\left[E^2 - 1 + \frac{2m}{R}\right]\ , \tag{30}$$

reveals an interesting fact. As the star's surface approaches the gravitational radius $dR/dt \to 0$. In fact it takes an infinite amount of time, t, for the star to reach $R = 2m$. An external observer therefore sees the star approach asymptotically to this surface.

This can be better understood in terms of light signals emitted from the star (see Fig. 2). As the star collapses, light emitted from the surface experiences an increasing gravitational field which tends to focus it. The closer the star is to $r = 2m$ the longer it takes for the emitted light to reach a distant observer O. In particular the light which is emitted at the gravitational radius $r = 2m$ takes an infinite amount of time to reach O. All subsequent light signals are focussed so strongly that they actually move towards $r = 0$. Therefore the external observer can have no knowledge of what happens to the star after it passes $r = 2m$ – this surface is therefore referred to as the black hole *event horizon*.

It is important to realize that the event horizon has no invariant local significance. It is a global construct, being the inner boundary of the causal past of an observer who remains outside of the black hole for an infinite time.

Thus we have two complementary views of gravitational collapse: for an external observer, the star appears asymptotically to approach the surface $r = 2m$ and never collapses

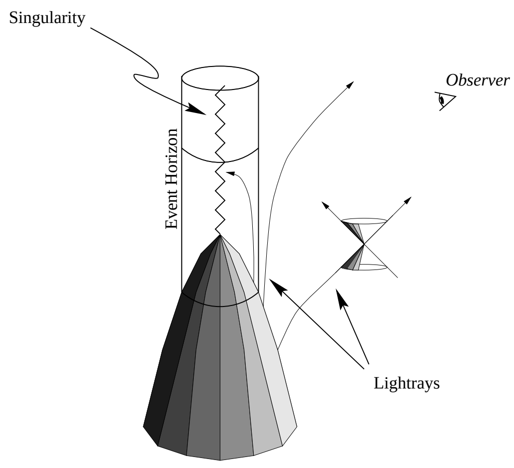

FIG. 2 Gravitational collapse of a spherical star to black hole. The event horizon is labelled $r = 2m$ in this diagram. Outgoing lightrays are shown leaving the surface of the star and reaching the external observer O. Those emitted after the star passes through the event horizon actually hit $r = 0$

further. Note that light from the star will be exponentially redshifted as its surface approaches $r = 2m$, making it quickly invisible to the external observer using optical devices. However the star actually passes, completely uninhibited, through the event horizon and reaches $r = 0$ where the curvature diverges. (In the Schwarzschild solution the Kretschmann invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48m^2/r^6 \to \infty$ as $r \to 0$.)

This picture remains qualitatively unchanged when the internal pressures of the star are non-zero (page 854 of (21)). Of course, no astrophysical collapse is expected to be spherically symmetric, so does the picture survive the inclusion of asphericities?

Spherical collapse with perturbations

During the 1960's many people argued that since black holes were exactly spherical they would not form in nature, however it gradually became clear that this was not true. An understanding of this fact was first achieved by numerical integration of the equations governing small perturbations of a spherical black hole (22; 23).

As the star collapses it radiates, by gravitational and other means, but certainly not fast enough to remove all the asphericities before it crosses the event horizon. Therefore one might think that an external observer should see a dirty black hole which is not exactly spherical. This is not the case. The method by which the black hole sheds this excess baggage was first elucidated in the work of Price (23).

In general the gravitational potential outside a black hole will scatter radiation which is present. Price noted that long wavelength radiation emitted from close to the black hole horizon ($2m < r < 3m$) is completely backscattered by the gravitational field and disappears down the black hole. Moreover, outgoing radiation emitted from close to the event horizon is redshifted by a very large factor as it moves to larger radii. Therefore information about the dirt on the black hole horizon which propagates outwards will become infinitely redshifted and is therefore completely backscattered down the black hole. It turns out that this backscattered radiation also interferes destructively with outgoing radiation, leading to a cleansing of the horizon and a Schwarzschild exterior at late times.

It might seem that the radiation that escaped from the collapsing star at earlier times (before it passed $r = 3m$) would simply escape to infinity. This is not the case though. It also gets scattered by the gravitational potential, leading to a diffusion of the radiation by a sequence of scatterings in the exterior of the black hole (see Gundlach *et al* (24) for a lucid exposition of this fact). Thus one finds a radiative tail of gravitational waves which decays with an inverse power law in external time. Along the event horizon of the black hole the stress-energy of the perturbations behaves like

$$\mathcal{F} \sim \sum_{l=2}^{\infty} P_l(\theta, \phi) v^{-4l-4} , \qquad (31)$$

in terms of a multipole expansion where v is external advanced time. This fact is important for the mass inflation scenario (18) (see chapter **??**).

The gravitational collapse of a spherically symmetric star, which produces a black hole, was first considered by Oppenheimer and Snyder (9) in 1939. Their work was poorly understood, however, going unnoticed for a long time.

For simplicity they examined the collapse of a spherical ball of dust. That is, the internal pressures and internal physics of the star were neglected. This assumption, which may appear unjustified at first sight, leads to a qualitatively correct picture of gravitational collapse (see (21) section 32.7) although this was not realized until long after their original work.

Since there is no pressure, the matter particles in the star follow geodesics of the spacetime; they are subject only to the gravitational force. Making one further assumption, that the radiation outside of the star is negligible, the external gravitational field is uniquely determined, by virtue of Birkhoff's theorem (21), to be the Schwarzschild solution: m denotes the total mass of the star and $d\Omega^2$ is the line element on the unit sphere.

III. LECTURE 4: COSMIC CENSORSHIP

Powerful global techniques (12) have been applied to prove that singularities cannot be avoided once a trapped surface[3] forms in collapsing matter (11). Unfortunately these results tell us nothing about the nature of the singularities which are encountered, nor do they tell us whether an event horizon will seal them off from external observers (i.e. if a black hole forms or not).

Penrose (25) has conjectured that any spacetime singularity will be surrounded by an event horizon – calling this phenomenon *cosmic censorship*. The proof of this conjecture remains elusive today. Indeed there is a mounting body of counter-examples (26) which severely constrain attempts to formulate the conjecture as a mathematical theorem. Most of these *naked singularities* are regarded as unrealistic, however they do suggest that only some quite limited form of cosmic censorship may be true (e.g. Thorne's hoop conjecture). Due to the great difficulty of this problem recent work has tended to focus either on the search for counter-examples, or on the derivation of conditions for the existence of trapped surfaces (27).

An interesting departure from these approaches has been given by Israel (28). He suggests that we may try to avoid the issues of singularities (initially) and ask when a trapped surface will necessarily lead to the formation of an event horizon. In fact he managed to prove a very beautiful theorem which states (roughly) that " a trapped 2-surface can be extended to a 3-cylinder that is and remains spacelike at least as long as it remains regular, thus it acts as a permanent one-way membrane for causal effects". Such results bolster our belief in cosmic censorship, and it is generally taken as a working hypothesis.

- Null fluid as an example of naked singularity demonstration $m = \lambda v$.
- EHC in spherical symmetry and statement in general

IV. LECTURE 5: COLLAPSE WITH ROTATION AND/OR CHARGE - INTERNAL STRUCTURE OF BLACK HOLES

The presence of rotation and/or charge in a collapsing star has a dramatic effect on the internal geometry of the black holes which form. The "no hair" theorems (10) prove that the only stationary electrovac black holes must belong to the Kerr-Newmann family. In view of our understanding of spherical collapse with perturbations we expect that the external gravitational field of a rotating black hole will settle down to a member of this family at late times. These solutions have a timelike singularity and suggest the possibility of travelling through the black hole into other universes. Since it is so difficult to analyse collapse with rotation, the Reissner-Nordstrom solution (which is the unique spherically symmetric electrovac solution of General Relativity (29)) is sometimes used as a model in which to test certain hypotheses, since these solutions have a similar causal structure to that of Kerr.

Gravitational collapse of charged dust was first considered by Novikov (30) and by de la Cruz and Israel (31). Their work shows that the gravitational collapse proceeds, in its external aspects, as it does in the uncharged case. The surface of the collapsing material reaches its gravitational radius in a finite proper time and an event horizon forms which seals off the subsequent evolution from external observers. The difference is that the contraction of the matter halts at some minimum radius (inside the event horizon), and then the dust begins to reexpand. In fact the evolution continues and the body emerges into another asymptotic region which is identical to, but distinct from, the one in which the collapse began. In so doing the matter has passed into a region of unpredictability. The future boundary of the domain of dependence for Cauchy data posed on the surface Σ is a null hypersurface – the Cauchy horizon – beyond which the evolution can no longer be uniquely determined. In particular, information can come out of the timelike singularity at $r = 0$ and affect the evolution in a completely unknown manner. Even to construct the analytically continued spacetime in Fig. IV.A.2 requires the assumption that nothing escapes from the singularity, and that the spacetime is vacuum (except for the electric field) in this region.

The Cauchy horizon is a highly pathological surface; small time-dependent perturbations originating outside the black hole undergo an infinite gravitational blueshift as they evolve towards the Cauchy horizon. This blueshift of infalling radiation gave the first indications that these solutions, which so well describe the exterior geometry at late times, may not describe the generic internal structure. Penrose (15) pointed this out some twenty five years ago, and since then linear perturbations have been analysed in detail (16). The divergence of the measured energy density of the perturbations at the Cauchy horizon led to the conjecture that a scalar curvature singularity would form once back-reaction was accounted for.

Since a generic collapse will almost certainly involve some rotation, an interesting problem is to understand what will be the structure inside the event horizon of such a black hole. This is the primary motivation for current investigations of black hole internal structure (17)-(20),(32)-(34). So far most of the analysis has been restricted to spherical symmetry where perturbations are modelled by lightlike dust or a scalar field. A first attempt by Hiscock to understand the internal geometry of a charged black hole with an influx of lightlike particles showed that an observer-dependent singularity was present along the Cauchy horizon. Poisson and Israel (18) included an outgoing flux of material across the Cauchy horizon and constructed a solution in which there is a null singularity, characterized by the divergence of the mass-function, along the Cauchy horizon. An outline of this work, which provides the launchpad for my contributions, is presented in chapter **??**. Poisson and Israel also argued that the physics behind their analysis is sufficiently general to believe

[3] A trapped surface T is a closed, spacelike two surface with the property that the two systems of null geodesics which meet T orthogonally converge locally in the future directions

that similar results should hold for generic collapse. Further detailed calculations seem to support this speculation that the singularity inside a generic black hole is null (20; 35).

The instability of the Cauchy horizon inside Reissner-Nordstrom black holes has been investigated by many authors (15; 16). In this lecture we summarise what is currently known about this instability and discuss the work of Hiscock (17), Poisson and Israel (18) and Ori (19) on the problem of the backreaction of perturbations near the Cauchy horizon.

The structure of the Reissner-Nordstrom solution is discussed in section **??** paying particular attention to the behaviour of timelike geodesics inside the event horizon. It is shown that an observer can pass through the black hole without encountering a singularity along his path.

A typical perturbing field Ψ decays in advanced time (which is infinite at the Cauchy horizon) according to an inverse power law, $\Psi \sim v^{-n}$ where n is a positive integer depending on the multipole order of the perturbation (23). However, a measurement of the field's rate of variation by a free-falling observer crossing the Cauchy horizon yields the infinite result

$$\Psi_{,\alpha} u^{\alpha} \simeq \Psi_{,v} \dot{v} \sim v^{-(n+1)} e^{\kappa_{\rm i} v} \tag{32}$$

where u^{α} is the four velocity of the observer (the dot denotes differentiation with respect to proper time) and $\kappa_{\rm i} = (m^2 - e^2)^{1/2}/r_i^2$ denotes the surface gravity of the inner horizon, m is the mass of the black hole, e its charge and $r_i = m - (m^2 - e^2)^{1/2}$ is the inner horizon radius. The same observer measures a flux of energy given essentially by the square of equation (XXX.32) which is even more divergent. Using a test field of lightlike dust we demonstrate this instability in section XXX.2.

The mechanism of this instability is the large blueshift occurring near to the Cauchy horizon. For reasonable observers residing outside the hole, $\dot{v} \simeq 1$, so that they require an infinite proper time to reach future null infinity ($v = \infty$). Internal observers require only a finite proper time to reach the Cauchy horizon, which implies that $\dot{v}$ will diverge as $v \to \infty$. As equation (XXX.32) shows, this divergence wins over the time decay of the perturbations and, as a result, the Cauchy horizon is said to be unstable.

The presence of this divergent flux also suggests that the backreaction of perturbations on the geometry will generate unbounded curvature along the Cauchy horizon. A preliminary investigation of the backreaction by Hiscock (17) showed that an observer-dependent singularity did form. The Hiscock model makes use of an exact solution of the Einstein field equations with null dust as a source. This model is presented in section XXX.3 and it is shown that the singularity which forms along the Cauchy horizon is characterized by the divergence of curvature in a parallel propagated orthonormal frame. This type of non-scalar curvature singularity is generally believed to be unstable, in the sense that slight perturbations will produce a scalar curvature singularity in its place. For charged, spherical black holes Poisson and Israel (18) have shown that a scalar curvature singularity forms at the Cauchy horizon when the above influx is accompanied by an outflux emitted from the collapsing star. Since they found the singularity to be characterized by a diverging mass function, they called it a *mass-inflation* singularity. We give a detailed discussion of the Poisson-Israel (18) scenario in section IV.D.

An exact solution, modelling the outflux as a thin shell of lightlike matter, was used by Ori to examine the nature of the mass-inflation singularity in some detail (19). He showed that an observer, who falls into the black hole, experiences finite tidal distortion at this singularity since the curvature is an integrable function of the observer's proper time.

For completeness the Ori model (19) limit of the presented solution is therefore taken in section XXX.5. It is then easy to obtain the relationship between proper time, along a timelike geodesic, and advanced time near to the singularity. Using this it is shown that an extended object undergoes only finite distortion up to the moment it crosses the Cauchy horizon. For this reason Ori suggested that the spacetime may be continued beyond the mass-inflation singularity. This is difficult to believe since there seems to be no (classical) mechanism which can drive curvatures back down to finite values beyond the Cauchy horizon. Herman and Hiscock (33) have argued that the question is not really relevant since anything approaching the Cauchy horizon will probably be "fried" by the ingoing blueshifted radiation.

A. The Reissner-Nordstrom solution

The unique spherically symmetric, charged black-hole solution of the Einstein field equations is the Reissner-Nordstrom solution. Using an advanced time coordinate v it has the line element

$$ds^2 = dv(2dr - f dv) + r^2 d\Omega^2 \qquad f = 1 - 2m/r + e^2/r^2 . \tag{33}$$

It is a static solution with timelike Killing vector $\xi^{\alpha} = \partial x^{\alpha}/\partial v$ outside the black hole. The global structure is different to that of Schwarzschild due to the presence of the charge. The Killing vector becomes null on two stationary lightlike hypersurfaces: the black hole event horizon $r = r_e$ and the inner (Cauchy) horizon $r = r_i$ determined by solving the quadratic $f = 0$, where the roots satisfy $0 < r_i < r_e$. The surface gravity is constant on each of the horizons being determined by the slope of f at that horizon, it is

$$\kappa_n = \frac{1}{2} \left| \partial_r f \right|_{r=r_n} \qquad n \in \{i, e\} . \tag{34}$$

1. Timelike geodesics

An analysis of timelike geodesics in this spacetime gives us some idea of the intrigueing global structure of this spacetime. The equation of a radial timelike geodesic can be reduced to

$$\dot{r}^2 = E^2 - f . \tag{35}$$

Let us suppose the energy satisfies $|E| > 1$, although a similar analysis can be done for the other cases. The right hand side of (35) can become zero at a finite radial value

$$r_b = \frac{-m + \sqrt{m^2 + (E^2-1)e^2}}{(E^2-1)} < r_i \,. \tag{36}$$

Therefore an observer who falls into the black hole decelerates until he reaches the finite radius r_b at which he comes to rest. Subsequently the same observer moves in the direction of increasing r and can return to arbitrarily large radii in a finite proper time. Thus it seems that it is possible to fall into a charged black hole, to avoid hitting the curvature singularity at $r = 0$ and to return to large radii again (see Fig. IV.A.2). Clearly this is very different to the behavior of geodesics in Scwharzschild spacetime (see section **??** in the Introduction).

2. Analytic continuation past the Killing Horizons

The system of coordinates (v, r) is singular on the ingoing sheet of the inner horizon: both r and v are constant there (v is actually infinite). They are also singular on the past event horizon. It is possible, however, to construct coordinates which are regular on each of these horizons in turn. As an example we do this for the inner black hole horizon. Introduce the coordinate

$$u = 2\int dr/f - v \,, \tag{37}$$

which is infinite on the outgoing sheet of the inner horizon. Moreover the coordinates (u, v) cover the region between the event and inner horizons (see Fig. IV.A.2). In terms of new coordinates (U, V) defined by

$$V = -e^{-\kappa_i v} \,, \tag{38}$$

$$U = -e^{-\kappa_i u} \,, \tag{39}$$

the line element (33) takes the form

$$ds^2 = \frac{f}{\kappa_i^2 UV}\, dUdV + r^2 d\Omega^2 \,. \tag{40}$$

In order to see that the metric is indeed regular on the inner horizon combine (37), (38) and (39) to get

$$UV = (r - r_i)G(r) \,, \tag{41}$$

where $G(r)$ is a function which has a non-zero value at $r = r_i$. Since we can write

$$f = \frac{(r - r_\mathrm{e})(r - r_\mathrm{i})}{r^2} \tag{42}$$

it is easy to see that the metric component $g_{UV} \to$ (constant) $\neq 0$ as $r \to r_i$. To analytically continue through the horizon, simply allow U and V to assume positive values. These coordinates then cover the entire black hole interior from the event horizon, $r = r_e$, down to the singularity at $r = 0$ – they are Kruskal coordinates for the inner horizon. One may analytically extend the spacetime through the other horizons in a similar manner, thus exhibiting the global structure of the manifold [see Fig. IV.A.2].

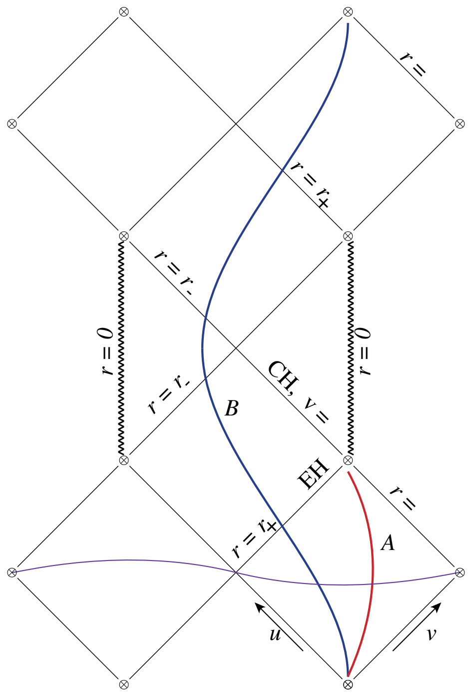

FIG. 3 Analytically extended Reissner-Nordstrom spacetime. EH is the event horizon $r = r_e$ and CH is the Cauchy horizon. A timelike observer follows the trajectory γ originating in our universe at O and continueing through the black hole to another asymptotic region.

B. Cauchy horizon instability

The instability of the Cauchy horizon of a Reissner-Nordstrom black hole is a direct consequence of the global structure of the spacetime manifold. The ingoing sheet of the inner horizon is the "continuation" of future null infinity ($\mathcal{I}^+$) inside of the black hole. An external observer requires an infinite amount of time to reach $\mathcal{I}^+$, however an observer who falls into the black hole requires only a finite amount of time to reach the Cauchy horizon. This observer will therefore see the entire history of the external universe flash quickly by. In particular the ratio of the proper time for an external observer to that of an internal observer will diverge as $v \to \infty$ implying that ingoing radiation will undergo an infinite gravitational blueshift at the Cauchy horizon. In a realistic situation an influx of radiation will always be present in the form of the radiative tail of the gravitational collapse which formed the black holes. We now examine a simple model of the gravitational perturbations(17; 18).

Superposed on the Reissner-Nordstrom background, we consider an influx of test radiation described by the stress-

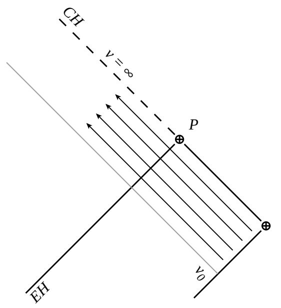

FIG. 4 A portion of the Reissner-Nordstrom spacetime, with infalling null radiation abutting the $v = \infty$ surface. The system of coordinates (v, r) covers the region below the $v = \infty$ line, and becomes singular on that line. Shown are the singularity ($r = 0$), the Cauchy horizon ($r = r_i$), the black hole horizon ($r = r_e$), and the asymptotic future null infinity.

energy tensor

$$T_{\alpha\beta} = \frac{L(v)}{4\pi r^2} l_\alpha l_\beta \tag{43}$$

and characterized by the luminosity function $L(v)$; the vector $l_\alpha = -\partial_\alpha v$ is tangent to ingoing radial null geodesics. We imagine radiation abutting the $v = \infty$ surface (see Fig. IV.B) and place restrictions on $L(v)$ by requiring that a radially free-falling observer in the exterior of the black hole measures the decaying radiative influx predicted by Price (23). Thus we write the luminosity function (as $v \to \infty$)

$$L(v) = \gamma(\kappa_i v)^{-p} \tag{44}$$

where $p \geq 12$ is an integer and γ is a dimensionless constant.

To see that the Cauchy horizon is unstable, consider a radially free-falling observer in the vicinity of the Cauchy horizon who measures the energy density $\rho_{\rm obs} = T_{\alpha\beta} u^\alpha u^\beta = T_{vv}\dot{v}^2$ for the influx of radiation. Normalizing the observer's four velocity so that $u^\alpha u_\alpha = -1$ we can write, using (35),

$$\dot{v} \simeq \frac{-2|E|}{f}\,, \tag{45}$$

as $v \to \infty$ and $f \to 0$. It can also be shown that near to the Cauchy horizon

$$f \simeq -2e^{-\kappa_i v} \tag{46}$$

along the observer's path. Thus $\dot{v} \simeq |E|\, e^{\kappa_i v}$ diverges at the Cauchy horizon, indicating that the observer requires only a finite proper time to reach $v = \infty$. Consequently, the observed energy density is

$$\rho_{\rm obs} \simeq \frac{\gamma\, E^2}{4\pi r_i^2}(\kappa_{\rm i} v)^{-p} e^{2\kappa_{\rm i} v} \tag{47}$$

from which we conclude that the energy density of the influx, as measured by a free-falling observer crossing the Cauchy horizon, will diverge as $v \to \infty$ indicating an instability of that horizon as Penrose (15) pointed out.

C. Charged Vaidya solutions

Turn this section into a discussion of the Ori solution – Patrick

Since an instability of the Cauchy horizon has been established, what is the effect of the divergent influx on the geometry? There exists an exact solution of the Einstein field equations with a stress-energy of the form (43). Hiscock (17) made a preliminary study of the backreaction of perturbations on the geometry using this solution. His results are summarized below.

The presence of the null dust simply makes the mass in (33) a function of advanced time v, so that it satisfies

$$\frac{dm}{dv} = L(v). \tag{48}$$

Therefore, enforcing the the inverse power law decay (44) on the influx the mass of the black hole

$$m(v) = m_1 - \frac{\gamma}{(p-1)\kappa_{\rm i}}(\kappa_{\rm i} v)^{-p+1} \tag{49}$$

increases to the value m_1 as $v \to \infty$. At first sight it seems that the radiation has not changed the geometry in any significant way. There continue to be two apparent horizons in the spacetime. The outgoing sheets of these horizons are no longer null, however the Cauchy horizon persists being located at $v = \infty$ and $r_{\rm i} = m_1 - \sqrt{m_1^2 - e^2}$. Of course it is the curvature, not the metric, which is important.

Since this solution includes material which streams along the Cauchy horizon, one expects that the curvature as measured by a free-falling observer will diverge there. Consider an orthonormal frame

$$e^\alpha_{(0)} \;=\; (\dot{v}, \dot{r}, 0, 0) = u^\alpha, \tag{50}$$
$$e^\alpha_{(1)} \;=\; (\dot{v}, f\dot{v} - \dot{r}, 0, 0), \tag{51}$$
$$e^\alpha_{(2)} \;=\; (0, 0, r^{-1}, 0), \tag{52}$$
$$e^\alpha_{(3)} \;=\; (0, 0, 0, (r\sin\theta)^{-1}), \tag{53}$$

where u^α is the observer's 4-velocity, and a dot represents differentiation with respect to proper time. It is straightforward to calculate the projected components of curvature in this frame. The three non-trivial components are

$$R_{(a)(B)(c)(D)} = \delta_{BD}\left(\eta_{ac}\left[\frac{1}{4}\partial_r^2 f + \frac{1}{2}\right] + 4\pi I_{ac}\,\rho_{\rm obs}\right) \tag{54}$$

where $a, c \in \{0,1\}$, $B, D \in \{2,3\}$, $\eta_{ab} = diag(-1,1)$ and I_{ab} is the two-dimensional matrix with all its entries equal to unity. As expected they diverge like the measured energy density (47) as the Cauchy horizon is approached. Despite this fact, it is possible to check that all the second order algebraic curvature scalars are bounded on the Cauchy horizon.

Can *any* curvature scalar diverge at the Cauchy horizon of this solution? For an influx given by (44), it seems not. In the coordinate system (v, r) the components of the metric and its inverse are bounded above and below (by zero) at the Cauchy horizon. In fact these components are C^∞ at the Cauchy horizon. Since an arbitrary curvature scalar is constructed from the metric and its derivatives, it is clear that all scalars must be bounded there also.

A singularity at which projected components of curvature diverge while scalars remain bounded has been called an *intermediate*, or *whimper* singularity by Ellis and King (52). Detailed studies indicate that such singularities are generally unstable and exist only in circumstances of high symmetry (53). It is therefore expected that a slight perturbation will induce a catastrophic plunge into a scalar curvature singularity. In this spherically symmetric case we now show that the inclusion of some outgoing null dust (proposed by Poisson and Israel (18)) is sufficient to completely destabilize the Cauchy horizon.

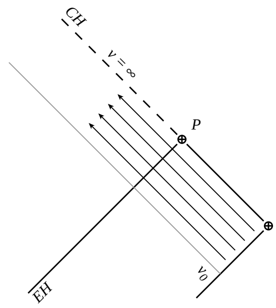

FIG. 5 The structure of the spacetime, with crossflowing null dust. EH is the event horizon and CH is the Cauchy horizon. The lines $u = u_0$ and $v = v_0$ at which the fluxes are switched on are also shown.

D. Backreaction of Spherical perturbations

Turn this into a discussion of the numerical results obtained by Brady/Smith, Burko and Hod/Piran – Patrick Our aim, in this section, is to construct an approximate solution which includes backreaction of perturbations (modelled by a lightlike influx and outflux) on the geometry and in this way to examine the Cauchy horizon singularity in detail. The general argument follows Poisson and Israel (18).

It is convenient to use null coordinates on the "radial" two spaces so that the spherical line element is

$$ds^2 = -\frac{2F}{r} du dv + r^2 d\Omega^2 \, , \tag{55}$$

where $F = F(u, v)$ and $r = r(u, v)$, and the coordinates are such that u is a retarded time and v an advanced time. The stress-energy tensor for a radial electromagnetic field is

$$E_\mu{}^\nu = e^2/4\pi r^4 \, \mathrm{diag}(-1, -1, 1, 1) \tag{56}$$

where e is the charge on the black hole. Poisson and Israel used crossflowing null dust to model the perturbations of the geometry, arguing that the large blueshift near to the Cauchy horizon should make the Isaacson (54) effective stress energy desciption valid for the ingoing radiation. They also pointed out that the nature of the outflux is not important, its only purpose is to initiate the contraction of the Cauchy horizon. The stress-energy tensor is

$$T_{\mu\nu} = \rho_{\mathrm{in}} l_\mu l_\nu + \rho_{\mathrm{out}} n_\mu n_\nu \, , \tag{57}$$

where $l_\mu = -\partial_\mu v$ and $n_\mu = -\partial_\mu u$ are radial null vectors pointing inwards and outwards respectively, and, ρ_{in} and ρ_{out} represent the energy densities of the inward and outward fluxes. Each term of (57) is independently conserved so that

$$\rho_{\mathrm{in}} = \frac{L_{\mathrm{in}}(v)}{4\pi r^2} \, , \qquad \rho_{\mathrm{out}} = \frac{L_{\mathrm{out}}(u)}{4\pi r^2} \, . \tag{58}$$

The functions $L_{\mathrm{in}}(v)$ and $L_{\mathrm{out}}(u)$ are to be determined by the boundary conditions, however, it is important to note that they have no direct operational meaning since they depend on the parametrization of the null coordinates.

The field equations can now be written as a pair of non-linear hyperbolic wave equations for r^2 and $\ln F$

$$(r^2)_{,uv} = F(e^2/r^2 - 1)/r \, , \tag{59}$$

$$(\ln F)_{,uv} = F(1 - 3e^2/r^2)/(2r^3) \, , \tag{60}$$

subject to the constraints

$$(r^2)_{,vv} - \frac{F_{,v}}{F}(r^2)_{,v} = -2L_{\mathrm{in}}(v) \, , \tag{61}$$

$$(r^2)_{,uu} - \frac{F_{,u}}{F}(r^2)_{,u} = -2L_{\mathrm{out}}(u) \, . \tag{62}$$

A comma denotes partial differentiation. Characteristic initial data for these equations is supplied along null rays $u = u_0$ and $v = v_0$.

Imagine that the inflow is turned on at a finite advanced time $v = v_0$ and the outflow is turned on at $u = u_0$. In the pure inflow (outflow) regime the solution is an ingoing (outgoing) Vaidya-Reissner-Nordström spacetime with mass function $m(v)[m(u)]$. The structure of the spacetime with cross-flow is shown in the Penrose diagram of Fig. IV.D.

We choose v to be standard advanced time far from the black hole, thus $v = \infty$ at the Cauchy horizon. We also use a convenient parametrization of the null coordinate u such that

$$F(u, v_0) = r_i, \tag{63}$$

where r_i is the radius of the static portion of the Cauchy horizon for $u < u_0$. It is now easy to obtain the behaviour of the radius along $u = u_0$ by solving the equations for a radial null geodesic in the Vaidya spacetime. A first integral of the geodesic equations is

$$\frac{dr}{dv} = \frac{f}{2} = \frac{1}{2}\left(1 - \frac{2m(v)}{r} + \frac{e^2}{r^2}\right) \tag{64}$$

where $m(v)$ is given by (49). This has the asymptotic solution

$$r|_{u=u_0} = r_{\rm i} + \frac{\gamma}{\kappa_i^2 r_i (p-1)}(\kappa_i v)^{-p+1}\left[1 + (p-1)(\kappa_i v)^{-1} + \ldots\right], \tag{65}$$

as $v \to \infty$. Now to determine $F(u_0, v)$ we substitute (65) into the constraint (61) to obtain (as $v \to \infty$)

$$(\ln F)_{,v} \simeq -\kappa_i\ , \tag{66}$$

which implies that

$$F(u_0, v) \simeq r_i e^{-\kappa_i v}\ . \tag{67}$$

Along the characteristic line $v = v_0$, the only remaining free datum is the radius of the two-sphere which must satisfy

$$(r^2)_{,uu} \ = \ -2L_{\rm out}(u)\ . \tag{68}$$

The behaviour of $L_{\rm out}(u)$ is not important so we simply assume that it is analytic in a neighbourhood of4 u_0.

For a given $L_{\rm out}(u)$ equations (63) and (65)-(68) complete the specification of the characteristic initial data. We can now proceed to find an approximate solution to the field equations.

It is crucial to obtain a serviceable approximation to the function $F(u, v)$ near to the Cauchy horizon, to this end we formally integrate (60) to

$$F = r_i\, g_1(u) g_2(v) \exp\left[\frac{1}{2}\int_{u_0}^{u}\int_{v_0}^{v} du' dv' \frac{F'}{(r')^5}\left\{(r')^2 - 3e^2\right\}\right]. \tag{69}$$

The functions $g_1(u)$ and $g_2(v)$ are determined by the initial data along the null rays $v = v_0$ and $u = u_0$. With the parametrization of the null coordinates defined by (63) we set $g_1(u) = 1$ and $g_2(v) = e^{-\kappa_i v}$.

In order to proceed, we must estimate the behavior of the integral in (XXX.69). It is expected that the major contribution to this integral should come from near to the Cauchy horizon since $v = \infty$ there. However, at least initially, we expect $r(u, v)$ to be a well behaved function with the *slow* contraction of ingoing null rays governed primarily by the outflux from the collapsing star. Thus we conclude that a good approximation to this metric coefficient is

$$F \simeq r_i\, e^{-\kappa_i v} e^{\mathcal{F}(u)}\ , \tag{70}$$

where

$$\mathcal{F}(u) = e^{-\kappa_i v_0}\left[\frac{r_i^2 - 3e^2}{2\kappa_i r_i^4}\right](u - u_0)\ , \tag{71}$$

near to $v = \infty$.

With this ansatz we can rewrite the constraints (61) and (62) as

$$(r^2)_{,uu} - \mathcal{F}_{,u}(r^2)_{,u} \ \simeq \ -2L_{\rm out}(u)\ , \tag{72}$$

$$(r^2)_{,vv} + \kappa_i\,(r^2)_{,v} \ \simeq \ -2L_{\rm in}(v)\ . \tag{73}$$

According to (70) we see that $F(u, v)$ goes to zero very rapidly near to the Cauchy horizon. Equation (59) can therefore be approximated by

$$(r^2)_{,uv} \simeq 0 \tag{74}$$

which gives the solution $r^2 \simeq R_{in}(v) + R_{out}(u)$. Substituting this into (72) and (73), and using the luminosity function defined in (44) we obtain the approximate solution

$$r^2 \simeq r_{\rm i}^2 + \frac{2\gamma}{\kappa_i^2(p-1)}(\kappa_i v)^{-p+1} - 2\int_{u_0}^{u} e^{\mathcal{F}'}\left[\int_{u_0}^{u'} e^{-\mathcal{F}''} L_{\rm out}(u'')du''\right]$$

Provided we limit our analysis to small $(u - u_0)$ the above approximation for $F(u, v)$ should be sufficient. It is worthwhile to notice that it can be separated into a function of u times a function of v. In particular by rescaling the coordinates we could make F constant on the Cauchy horizon.

It should also be noted that

$$\int_{u_0}^{u} e^{\mathcal{F}'}\left[\int_{u_0}^{u'} e^{-\mathcal{F}''} L_{\rm out}(u'')du''\right] du' \simeq \int_{u_0}^{u}\left[\int_{u_0}^{u'} L_{\rm out}(u'')du''\right] du' \tag{76}$$

when $u - u_0$ is small. Thus as stated above the contraction of the ingoing lightrays near the Cauchy horizon is governed primarily by the outflux from the star.

1. The mass function

For a spherically symmetric system it is possible to introduce a geometrically defined mass function $m(x^\alpha)$ via the gradient of the area of the two spheres:

$$g^{\alpha\beta}\nabla_\alpha r \nabla_\beta r = 1 - \frac{2m}{r} + \frac{e^2}{r^2}\ , \tag{77}$$

where ∇ indicates the four dimensional covariant derivative. At infinity it corresponds to the ADM mass (see for example page 293 of (10)). Furthermore, it is equivalent to Hawking's quasi-local mass (55). It also acquires operational meaning in the spherically symmetric case since it determines the Weyl curvature $|\Psi_2| = (m - e^2/r)/r^3$.

Using (59), (62) and (77) it is easy to show that the mass function $m(x^\alpha)$ satisfies

$$m_{,v} = -\frac{L_{in}(v)(r^2)_{,u}}{2F} \simeq \frac{\gamma(\kappa_i v)^{-p} e^{\kappa_i v}}{r_i} \int L_{out}(u) e^{-\mathcal{F}} du \, . \tag{78}$$

Thus the mass in the crossflow region ($u > u_0$ and $v > v_0$)

$$m(u,v) \simeq \frac{\gamma}{\kappa_i}(r_i\kappa_i v)^{-p} e^{\kappa_i v} \int L_{out}(u) e^{-\mathcal{F}} du \, , \tag{79}$$

inflates to infinity exponentially in advanced time v. This is the result obtained by Poisson and Israel (equation (4.17) in (18)).

2. The strength of the mass inflation singularity

The strength of the mass-inflation singularity is best discussed in terms of a simple model of a black hole interior considered by Ori (19). He replaced the continuous outflux by a delta function source at $u = u_0$. This has the advantage that the spacetime continues to be described by an ingoing Vaidya solution for $u > u_0$. In this region the mass function does not approach a finite asymptotic value, rather it inflates to infinity on the Cauchy horizon.

Writing $L_{out}(u) = a\,\delta(u - u_0)$ in (75) and (79) the line element of Ori's solution is

$$ds^2 \simeq -\frac{2r_i e^{\mathcal{F}} e^{-\kappa_i v}}{r} du dv + r^2 d\Omega^2 \, , \tag{80}$$

where

$$r^2 \simeq r_{\rm i}^2 + \frac{2\gamma}{\kappa_i^2(p-1)}(\kappa_i v)^{-p+1} - 2a(u-u_0) \, . \tag{81}$$

The mass function (79) is

$$m(v) \simeq \frac{a\gamma}{r_i\kappa_i}(\kappa_i v)^{-p} e^{\kappa_i v} + \dots \, , \tag{82}$$

for $u > u_0$, in complete agreement with (19).

Although there is a scalar curvature singularity in this model at the Cauchy horizon, Ori suggested that the tidal distortion of extended objects approaching the singularity is physically more relevant as a means of determining the strength of the singularity. Thus a singularity is weak (according to Ori) if an extended object undergoes only finite tidal distortion all the way up to the singularity.

The tidal forces experienced by an observer are proportional to the projected components of curvature in an orthonormal frame $\{e^\alpha_{(a)}\}$ parallel propagated along his path (see for example (21) page 860). Choosing $u^\alpha = e^\alpha_{(0)} = (\dot{u}, \dot{v}, 0, 0)$ as the timelike vector, we have

$$\frac{2F}{r}\dot{u}\dot{v} = 1 \, , \tag{83}$$

where a dot represents diferentiation with respect to the observers proper time τ. The geodesic equations and this first integral imply that

$$\ddot{v} \simeq \kappa(\dot{v})^2 \quad \text{as } \; v \to \infty \, , \tag{84}$$

which is easily integrated. If we choose $\tau = 0$ on the Cauchy horizon the relation between it and the advanced time is

$$\tau \simeq \text{const} \times e^{-\kappa_i v} \qquad \text{as } \; v \to \infty \, . \tag{85}$$

Assuming that internal stresses of the body can be neglected to a first approximation, the tidal distortion is given by twice integrating the projected curvature. From equation (54) it is easy to see that the leading terms in the tidal forces are

$$R_{\dots} \sim \frac{L_{in}(v)}{r^2}(\dot{v})^2 \sim |\ln\kappa_i\tau|^{-p}(\kappa_i\tau)^{-2} \, . \tag{86}$$

Integrating this function twice with respect to τ gives a bounded quantity as $\tau \to 0$, indicating that the singularity is weak according to the definition above.

In fact using τ as a coordinate the metric can be recast in the form

$$ds^2 \simeq -\frac{2r_i e^{\mathcal{F}}}{r} du d\tau + r^2 d\Omega^2 \, , \tag{87}$$

where

$$r^2 \simeq r_{\rm i}^2 + \frac{2\gamma}{\kappa_i^2(p-1)}|\ln(\kappa_i\tau)|^{-p+1} - 2a(u-u_0) \, . \tag{88}$$

Thus the metric and its inverse are bounded at the Cauchy horizon and $\sqrt{-g}$ is non-zero there. It is this fact that led Ori to suggest that spacetime might be continued beyond the mass-inflation singularity (19).

E. Conclusion

In this chapter we have presented a detailed analysis of the Cauchy horizon instability for Reissner-Nordstrom black holes. In the test field approximation, where a null dust was used to model perturbations on the fixed background, it was shown that the Cauchy horizon is unstable due to an infinite gravitational blueshift of infalling radiation. This was first pointed out by Penrose (15) some twenty five years ago, and linearized perturbations have been analysed by many authors (16).

An approximate solution to the Einstein field equations coupled to cross-flowing null dust was also presented. As pointed out by Poisson and Israel (18) it indicates that the mass function diverges exponentially in external advanced time, which is infinite on the Cauchy horizon. Unlike the Hiscock model (17), where no scalar curvature diverges, the presence of a continuous outflux $L_{\rm out}(U)$ means that the Kretschmann invariant diverges like the locally measured energy density

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \sim 16 e^{-2\kappa_i v} e^{\mathcal{F}(u)} \frac{L_{\rm out}(u) L_{\rm in}(v)}{r_i^4} \tag{89}$$

As pointed out by Ori (19), the presence of the outflux from the star does not enhance the singularity enough to cause infinite tidal distortion of an observer reaching the Cauchy horizon. Ori's conclusion that spacetime can perhaps be continued beyond the mass-inflation singularity has stirred some debate (32; 33). In any case it is difficult to see how the infinite curvature might be contained in a thin layer at the Cauchy horizon followed by a region of bounded curvature.

The discussion in this chapter has been limited to spherical symmetry. Poisson and Israel (18) argued that the presence of asphericities should not significantly change the mass-inflation scenario. Indeed, asymptotic and perturbative analyses (19) of the singularity inside more realistic black holes suggest that this should be the case. Further investigation of aspherical models of black hole interiors is currently under way (56), and a progress report on this work is presented in chapter **??**.

Acknowledgments

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APPENDIX A: Two-dimensionally covariant formalism for spherical symmetry

Most of the results presented in these lectures will be based in spherical symmetry, for this reason it is useful to have a formalism to deal with Einstein's equations for an arbitrary matter source in spherical symmetry. The results here were obtained in this two dimensional covariant form by Poisson and Israel (18).

The spacetime metric is written as

$$ds^2 = g_{ab}dx^a dx^b + r^2 d\Omega^2 \tag{A1}$$

where g_{ab} is the metric on the radial two sections, r^2 is a scalar function of x^a and $d\Omega^2$ is the line element on the unit two sphere. Following PI notation such that four dimensional quantities are indicated by a superscript 4, while two dimensional quantities have no superscript, the Riemann curvature is

$$\begin{aligned} {}^4R_{abcd} &= R_{abcd} && \text{(A2)} \\ {}^4R_{a\theta c\theta} &= \sin^{-2}\theta\ {}^4R_{a\phi\, c\phi} = -r r_{;ab} && \text{(A3)} \\ {}^4R_{\theta\phi\,\theta\phi} &= r^2\sin^2\theta\,(1 - r^{,a}r_{,a}) && \text{(A4)} \end{aligned}$$

where (,) represents partial differentiation, and (;) is the covariant derivative associated with g_{ab}.

The Ricci tensor is

$$\begin{aligned} {}^4R_{ab} &= R_{ab} - 2r_{:ab}/r && \text{(A5)} \\ {}^4R_{\theta\theta} &= \sin^{-2}\theta\ {}^4R_{\phi\phi} = 1 - (r\Box r + r^{,a}r_{,a})\,, && \text{(A6)} \end{aligned}$$

and the Ricci scalar

$${}^4R = R + 2(1 - 2r\Box r - r^{,a}r_{,a})/r^2\,. \tag{A7}$$

Now introduce the scalars $m(x^a)$ and $\kappa(x^a)$

$$\begin{aligned} g^{ab}r_{,a}r_{,b} &:= f := 1 - 2m/r + e^2/r^2\,, && \text{(A8)} \\ \kappa &:= -\frac{1}{2}\partial_r f := -(m - e^2/r)/r^2\,, && \text{(A9)} \end{aligned}$$

The Einstein field equations

$$G_{\alpha\beta} = 8\pi(E_{\alpha\beta} + T_{\alpha\beta}), \tag{A10}$$

where $E^{\alpha}_{\beta} = XXXX$ is the stress-energy of a spherical symmetric electromagnetic field, can now be written as

$$\begin{aligned} r_{;ab} + \kappa g_{ab} &= -4\pi r(T_{ab} - g_{ab}T)\,, && \text{(A11)} \\ R - 2\partial_r\kappa &= 8\pi(T - 2P)\,. && \text{(A12)} \end{aligned}$$

Here T_{ab} is the two dimensional submatrix of the stress-energy tensor $T_{\alpha\beta}$, $T = g^{ab}T_{ab}$ and $P = T^{\phi}{}_{\phi} = T^{\theta}{}_{\theta}$. Therefore $T_{\alpha\beta}$ is the stress-energy of all other forms of matter which are included as source. (The uncharged case can be obtained by setting $e = 0$ in the above equations.)

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